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Singularity and Dynamics on Discontinuous Vector Fields

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MONOGRAPH SERIES ON NONLINEAR SCIENCE AND COMPLEXITY

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Preface

This book is about dynamics, which is a very old topic with a long history. But it is very important and necessary for us to further understand and develop dynamics theory for describing complexity and variety in nature. Because Newton presented three motion laws for dynamics in 1686, he systematically developed Calculus for exploring the implications of three laws. Newton's Laws successfully have provided a tool for us to predict the behavior of mechanical objects and have led to the unbelievable development and progress in science and technology. Based on the Newton's development, Lagrange further developed a variational principle to reformulate the Newtonian mechanics, and such a variational principle is used to analyze the stability of dynamic systems. In addition, Hamilton used momentum–displacement phase space to describe the Newton's mechanics and discuss the motion symmetry in phase space. In 1892, Poincaré demonstrated that the inherent characteristics of the motion in the vicinity of unstable fixed points of the nonlinear oscillation systems may be stochastic under the regular applied forces. Chaos in continuous, nonlinear dynamical systems has been extensively investigated in recent decades. All the aforementioned theories are based on the Lipschitz condition. However, due to the complexity of described problems, a global system consists of many uniquely-continuous subsystems. Each subsystem with a unique continuity possesses its own dynamical properties different from the neighbored subsystems. Further, the motion from a subsystem to another subsystem requires the passability conditions or transport laws. The diversified properties of all the subsystems make the richer dynamical behaviors than continuous dynamical systems only with the unique continuity. Such discontinuous dynamical systems exist everywhere in engineering and science. So far, many researchers tried to smooth discontinuous systems, and then theory and techniques in continuous dynamical system were used to solve such discontinuous dynamical systems. Therefore, in this book, a different point of view would like to be adopted to look into dynamical behaviors in discontinuous dynamical systems.

This book focuses on dynamics on discontinuous vector fields. To describe the discontinuous dynamic systems, the novel concepts are presented through Calculus in the entire book. This book consists of eight chapters. Chapter 1 provides a brief literature survey for current researches in this area. Chapter 2 introduces the necessary concepts for discontinuous dynamical systems, and the grazing flows in the vicinity of the discontinuous boundary will be presented. In Chapter 3, the sliding flow is discussed and the switching bifurcations between passable and nonpassable flows will be discussed. Chapter 4 presents the transversal tangential flows and bouncing flows in discontinuous dynamic systems. In Chapter 5, the real and imaginary flows in discontinuous dynamic systems are introduced for passable and non-passable flows. Chapter 6 presents discontinuous systems with flow barriers, and the sliding flows and the corresponding switching bifurcations are discussed. The transport laws and mapping dynamics are discussed in Chapter 7. Finally, flow symmetry and strange attractor fragmentation are presented in Chapter 8.

I would like to thank my students (Brandon C. Gegg, Lidi Chen and Patrick Zwiegart Jr., etc.) to apply the new concepts to practical problems in mechanical systems. This book is dedicated to people who love to generate the original theory. Finally, I also dedicate to my wife (Sherry X. Huang) for support and to my lovely children (Yanyi, Robin, and Robert) for their happiness to stimulate my inspiration.

Albert C.J. Luo
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Contents

Chapter 1

Introduction

With computer invention and computation speed enhancement, the complicated scientific computation becomes possible and more efficient. Further, one has tried to develop accurate models to describe the complex systems in science and engineering. Hence, in recent decades, chaos and fractals were discovered through nonlinear models even for dynamical systems with a unique continuity. Owing to the complexity of described problems, a global system consists of many uniquely-continuous subsystems. Each subsystem with a unique continuity possesses its own dynamical properties at least different from the neighbored subsystems. Further, the motion from a subsystem to another subsystem requires the passability conditions or transport laws. The diversified properties of all the subsystems make the richer dynamical behaviors than continuous dynamical systems only with unique continuity. Such discontinuous dynamical systems exist everywhere in engineering. However, each uniquely-continuous subsystem with unique-continuity should be investigated by the continuous dynamical system theory. Therefore, the brief introduction of continuous dynamical systems will be given first, and then dynamical systems with discontinuity will be presented. The survey for two typical mechanical problems will be presented herein to demonstrate the importance and significance of investigations on discontinuous dynamical systems. Finally, the book layout will be presented. The summarization of all chapters of the main body of this book will be given.

1.1. Smooth dynamics

It is quite difficult to give analytical solutions of dynamical responses in nonlinear dynamical systems. With computers expansively used in engineering and science, numerical simulations as a useful tool play a very important role on obtaining dynamical responses in nonlinear dynamics, which help one understand complexity in nature. However, the current digital computation is very passive, and the approximation-based algorithms cannot provide all possible complicated responses existing in dynamical systems, such as regular and chaotic motions caused by bifurcation and grazing etc. This is also partially because of the singularity of solutions for complicated dynamical responses. Numerical simulations may find one of all possible solutions, but this solution may not belong to the same solution branch because the singularity will lead to jumping or catastrophe phenomena. Those phenomena often confuse people. Thus one has been very interested in digging out the mechanisms of such nonnormal phenomena. To model complicated dynamical problems in science and engineering, it is necessary and important to investigate the dynamics of complex dynamical systems. The complex dynamical systems, consisting of many continuous and discrete subdynamical systems, exist almost everywhere in engineering, such as hybrid systems in automatic control, computer science and chemical process plants; dynamical systems for large data transmission and management in Internet; piecewise smooth dynamical systems in electrical and mechanical engineering, etc. Therefore, this book will plan to present some initial ideas to investigate the global dynamics of complex dynamical systems with discontinuities.

Before complex systems with multiple discontinuities are discussed, a brief history of continuous dynamical system is presented. Since Newton in the 17th century proposed the three motion laws based on the qualitative mechanics summarized by Kepler and Galileo, etc., the quantitative theory of mechanics (smooth dynamics) had been systematically developed by Newton, Euler, Lagrange, Laplace and Hamilton. The stability of dynamical systems was one of the main concerns, which was analyzed through the series expansion techniques. In the end of the 19th century, Poincaré (1892) developed a qualitative, geometric method to investigate the stability of dynamical systems through modern differential geometry and topology instead of the traditional calculus, differential equations and variational theory. The Poincaré method can provide qualitative analysis (or global analysis) for stability, bifurcation and chaos. Poincaré (1892) also introduced the mapping concept for determination of periodic responses in nonlinear dynamics. Such a mapping is termed the Poincaré mapping. The mapping sets accompanying with the Poincaré mapping is termed the Poincaré mapping section or surface. For an N-dimensional autonomous system, the Poincaré mapping section is selected as an $(N-1)$-dimensional surface transversal to the closed orbit. When a periodically-driven, N-dimensional *continuous* system is investigated, the Poincaré mapping section is often constructed by an N-dimensional set of responses in phase space. In beginning of last century, one used the Poincaré mapping method to periodic motion and stability (e.g., Birkhoff, 1927; Stoker, 1950; Corddington and Levinson, 1955; Smale, 1967). Since the middle of last century, the Poincaré mapping has been used to demonstrate chaotic motions numerically (e.g., Henon and Heiles, 1964; Ueda, 1980). The mapping has been extensively used for investigation of complicated motions in nonlinear systems. Through numerical simulations, one observed many motion characteristics which cannot be imagined through traditional, analytical tools. To investigate the complex motion of the uniquely-continuous dynamical systems, the catastrophe and bifurcation theories have further been developed to explain strange phenomena in nonlinear dynamic systems (e.g., Arnold, 1989; Guckenheimer and Holmes, 1983). In this book, such existing theories for bifurcations and chaos in continuous dynamical systems will not be discussed. Consider a smooth dynamical system in space $\Re^{n\times m}$

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \tag{1.1}$$

where the vector function $\mathbf{f} \in \Re^n$, and state and input variable vectors are $\mathbf{x} \in \Re^n$ and $\mathbf{u} \in \Re^m$, respectively. In smooth dynamical systems, the sufficient condition for the existence of a solution for every initial state $\mathbf{x}(t_0)$ and input vector $\mathbf{u}(t)$ is that the vector function $\mathbf{f}(\mathbf{x}, \mathbf{u}, t)$ is continuous in a given domain $\Omega \subset \Re^n$. However, this condition cannot guarantee the uniqueness of solution. Therefore, the following Lipschitz condition is used for guaranteeing the existence and uniqueness of the solution for the system in Eq. (1.1):

$$\left\|\mathbf{f}(\mathbf{x}, \mathbf{u}, t) - \mathbf{f}(\tilde{\mathbf{x}}, \mathbf{u}, t)\right\| \leqslant K\|\mathbf{x} - \tilde{\mathbf{x}}\| \tag{1.2}$$

for all $\mathbf{x}$ and $\tilde{\mathbf{x}}$ in the domain $\Omega \subset \Re^n$ and all time t in a certain interval, where K is a constant and $\|\cdot\|$ represents a vector norm. Most of the existing theories in dynamics are based on the Lipschitz condition (e.g., Poincaré, 1892; Birkhoff, 1927). Indeed, those theories are widely used in science and engineering. However, one wants to develop the expected dynamic behavior to satisfy specified requirements. Hence discontinuous constraints destroying the Lipschitz conditions are added to dynamic systems. Because of this reason, the established dynamical system theories based on the Lipschitz condition are not adequate for such nonsmooth dynamical systems. Levinson (1949) introduced the simplified, piecewise linear models to investigate the periodically forced Van der Pol's equation. Based on this piecewise linear model, Levinson found inifinitely many periodic solutions which could not be perturbed away. This finding is far earlier than the Smale's horseshoe in Smale (1963, 1967) from a topological point of view. The further results about the piecewise linear model of the periodically forced Van der Pol oscillator can be referred to Levi (1978, 1981). In Engineering, for instance, smooth linear dynamical systems with periodic impacting (e.g., Masri and Caughey, 1966; Luo and Han, 1996) have complicated dynamical behaviors which are unpredictable from the traditional dynamical theories. The Lipschitz condition is very strong for practical dynamical problems, and many dynamical systems cannot satisfy such a condition. To overcome this difficulty, a theory for discontinuous dynamical systems should be developed further.

1.2. Nonsmooth dynamics

The early investigation of discontinuous systems in mechanical engineering can be found in the 30's of last century (e.g., Den Hartog 1930, 1931; Den Hartog and Mikina, 1932). Masri and Caughey (1966) investigated the stability of the symmetrical period-1 motion of a discontinuous oscillator. Masri (1970) gave the further, analytical and experimental investigations on the general motion of impact dampers. The unsymmetrical motion was observed, and the rigorous stability analysis was conducted as well. Since the discontinuity exists widely in engineering and control systems, Utkin (1978) presented sliding modes and the corresponding variable structure systems, and the theory of automatic control systems described with variable structures and sliding motions was also developed in Utkin (1981). Further, Filippov (1988) developed a geometrical theory of the differential equations with discontinuous right-hand sides, and the local singularity theory of the discontinuous boundary was discussed qualitatively. Ye et al. (1998) discussed the stability theory for hybrid systems. From geometrical points of view, Broucke et al. (2001) investigated structural stability of piecewise smooth systems. So far, an efficient method to model such nonsmooth dynamical systems has not been developed yet. For instance, the linear impacting oscillators cannot be fully understood as one of the simplest discontinuous systems (e.g., Senator, 1970; Bapat et al., 1983; Shaw and Holmes, 1983a; Luo, 1995, 2002; Han et al., 1995). Another typical example in engineering is piecewise smooth linear systems. Shaw and Holmes (1983b) used mapping techniques to investigate the chaotic motion of a piecewise linear system with a single discontinuity. Natsiavas (1989) numerically determined the periodic motion and stability for a system with a symmetric, tri-linear spring. Nordmark (1991) introduced the grazing mapping to investigate nonperiodic motion. Kleczka et al. (1992) investigated the periodic motion and bifurcations of piecewise linear oscillator motion, and numerically observed the grazing motion. Leine and Van Campen (2002) investigated the discontinuous bifurcations of periodic solutions through the Floquet multipliers of periodic solutions. The analytical prediction of periodic responses of piecewise linear systems was presented (e.g., Luo and Menon, 2004; Menon and Luo, 2005). Normal formal mapping for piecewise smooth dynamical systems with/without sliding were discussed (e.g., di Bernardo et al., 2001, 2002). Kunze (2000) presented a mathematical background of a nonsmooth dynamical system with friction. Popp (2000) pointed out that: (i) solution methods need to be improved, (ii) efficient methods for stability and bifurcation are required to develop, and (iii) the attractor characteristics need to be reconstructed. Luo (2005a) developed a local theory for nonsmooth dynamical systems on connected domains. This theory was used to piecewise linear systems (e.g., Luo, 2005b, 2006) and friction-inducted oscillator (e.g., Luo and Gegg, 2006a, 2006b). Luo and Chen (2005a, 2006) investigated the periodic and chaotic motion of the piecewise linear systems with impacting. The concept for the strange attractor fragmentation in discontinuous systems was proposed in Luo (2005c). Two typical mechanical discontinuous problems will be discussed as follows.

In mechanical engineering, the friction contact between two surfaces of two bodies is an important connection in motion transmissions. The material plastic flows in plastic deformation are a kind of friction behavior. In the material cutting, the frictions both from the plastic deformation and between the tool and workpiece are an important source to cause the self-excited vibration. Therefore, the understanding of the friction effects on oscillations is very significant to determine the dynamics of machines in engineering. In existing investigations, the focuses lied on the responses of dynamical systems with friction. Den Hartog (1930) investigated the nonstick periodic motion of the forced linear oscillator with Coulomb and viscous damping. Levitan (1960) investigated the existence of periodic motions in a friction oscillator with the periodically driven base. Hundal (1979) investigated the frequency–amplitude response of such an oscillator. Shaw (1986) investigated the stability for such a nonstick, periodic motion through the Poincaré mapping and the local stability theory. Feeny (1992) investigated analytically the nonsmoothness of the Coulomb friction oscillator. Feeny and Moon (1994) presented chaotic dynamics of a dry-friction oscillator experimentally and numerically. Further, Feeny (1996) systematically presented the nonlinear dynamical mechanism of oscillators with stick-slip friction. Hinrichs et al. (1997) presented the nonlinear phenomena in an impact and friction oscillator under external excitation (see also Hinrichs et al., 1998). The stick and nonstick motions were observed, and chaos for a nonlinear friction model was presented. From the previous investigations, the friction oscillators under periodical excitations exhibit very rich dynamical characteristics. Natsiavas (1998) developed an algorithm to numerically determine the periodic motion and the corresponding stability of piecewise linear oscillators with viscous and dry friction damping (see also Natsiavas and Verros, 1999). Leine et al. (1998) obtained the limit cycles of the nonlinear friction model by the shooting method. Virgin and Begley (1999) determined the grazing bifurcation and attraction basin of an impact-friction oscillator through the interpolated cell mapping method. Ko et al. (2001) investigated the friction-induced vibrations with and without external excitations. Andreaus and Casini (2002) presented the closed form solutions of the Coulomb friction-impact model without external excitations. Thomsen and Fidlin (2003) gave an approximate, analytical amplitude for stick-slip vibration with nonlinear friction model. Kim and Perkins (2003) investigated nonsmooth stick-slip oscillator through the harmonic balance/Galerkin method. Pilipchuk and Tan (2004) investigated the friction-induced vibration of a two-degree-of-freedom mass-damper–spring system interacting with a decelerating rigid strip. Li and Feng (2004) investigated the bifurcation and chaos in the friction-induced oscillator with a nonlinear friction model.

Pfeiffer (1994) used an impact and friction model to investigate the unsteady process in machines. To investigate the contact problem between the two surfaces of two bodies, Pfeiffer and Glocker (1996) developed the theory for multi-body dynamics with unilateral contacts by using analytical dynamics. Pfeiffer (2000) gave a brief introduction of the theory of the unilateral multibody dynamics, and presented some typical applications (see also Pfeiffer, 2001). From a mathematical point of view, Filippov (1964) investigated the motion in the Coulomb friction oscillator and presented differential equations with discontinuous right-hand sides. To determine the sliding motion along the discontinuous boundary, the differential inclusion was introduced via the set-valued analysis, and the existence and uniqueness of the solution for such a discontinuous differential equation were discussed. The detailed discussion of such discontinuous differential equations can be referred to Filippov (1988). However, the Filippov's theory mainly focused on the existence and uniqueness of the solutions for nonsmooth dynamical systems. The local singularity caused by the separation boundary was not discussed. Such a differential equation theory with discontinuity is still difficult to use for determining the complexity of nonsmooth dynamical systems. In addition to a general theory for the local singularity of nonsmooth dynamical systems on connectable domains in Luo (2005a), the imaginary, sink and source flows were presented in Luo (2005d) to determine the sliding and source motions in nonsmooth dynamical systems. Luo and Gegg (2006a) used the local singularity theory to develop the force criteria for the harmonically driven linear oscillator with dry-friction. For such an oscillator, the periodic motion was predicted analytically through the mapping structures, and the stick and nonstick motions were observed in Luo and Gegg (2006b). The methodology for mapping structures for periodic motion can be systematically presented in Luo (2005e), and such an idea was used to determine the periodic and chaotic motions (e.g., Luo and Menon, 2004; Menon and Luo, 2005). Furthermore, Luo and Gegg (2006c, 2006d) used the nonsmooth theory of Luo (2005a, 2005d) to systematically investigate the mechanism of grazing and stick motions in such a friction oscillator. The sufficient and necessary conditions were obtained for the existence of the grazing and stick motions. Through such an investigation, the traditional eigenvalue analysis in discrete dynamical systems is not useful for the motion switching at the discontinuous boundary.

In mechanical engineering, the reduction of the noise and vibration in gear transmission systems is an unsolved problem. Such noise and vibration cause the inefficiency of the gear transmission systems. Thus, ones have put more efforts to enhance the efficiency of gear transmission systems and reduced the vibration and noise. The early investigations of the gear transmission systems focused on the mesh geometries, kinematics and strength of teeth (e.g., Buckungham, 1931, 1949). For low-speed gear systems, the linear model was developed, and this linear model gives a reasonable prediction of gear-teeth vibrations. However, the vibrations and noises in high-speed transmission systems become serious. The linear vibration model cannot provide the adequate prediction of such vibrations and noises. Therefore, in recent decades, both the piecewise linear model and the impact model were developed to find the origin of the vibration and noise (e.g., Ozguven and House, 1988). The impact model was developed for regular and chaotic motions in gear box (e.g., Pfeiffer, 1984; Karagiannis and Pfeiffer, 1991). For such a purpose, the gear transmission systems are also modeled by the piecewise linear system. The early studies of the piecewise linear systems can be found in 1931. For instance, Den Hartog and Mikina (1932) gave the closed-form solution of the periodic symmetric motion of the piecewise linear system without damping. The piecewise linear system was used to model the gear transmission system (e.g., Comparin and Singh, 1989; Kahraman and Singh, 1990; Theodossiades and Natsiavas, 2000). The nonlinear responses based on such a model were presented. To obtain the nonlinear responses in the piecewise linear system, Wong et al. (1991) used the incremental harmonic balance method to obtain the periodic motions. Kim and Noah (1991) gave the stability and bifurcation analysis through the harmonic balance method (see also Kim, 1998). To model vibrations in gear transmission systems, recently, Luo and Chen (2005a) investigated the simplest periodic motion of the piecewise linear, impacting system under a periodical excitation. The corresponding grazing of periodic motions was observed. The chaotic motion for this system was numerically simulated. The grazing mechanism of the strange fragmentation of such an oscillator was discussed (Luo and Chen, 2006). For such an oscillator, not only the simple periodic motion exists, but also many complicated periodic motions are observed in practice. The switching from the two periodic motions was observed, and the asymmetric periodic motions exhibit the period-doubling bifurcation rather than the symmetric one.

1.3. Book layout

This book consists of eight chapters. Chapter 1 provides a brief literature survey and presents basic contents per each chapter. Chapter 2 will introduce the basic concepts for discontinuous dynamical systems, and the grazing flows in the vicinity of the discontinuous boundary will be presented. In Chapter 3, the sliding flow will be discussed and the switching bifurcations between passable and nonpassable flows will be discussed. Chapter 4 will present the transversal tangential flows and bouncing flows in discontinuous dynamical systems. In Chapter 5, the real and imaginary flows will be introduced to describe passable and nonpassable flows in discontinuous dynamical systems. Chapter 6 will present discontinuous systems with flow barriers, and the sliding flows and the corresponding switching bifurcations will be discussed. The transport laws and mapping dynamics will be discussed in Chapter 7. Finally, flow symmetry and strange attractor fragmentation will be presented in Chapter 8. The detailed descriptions of chapters for the main body of this book are given as follows.

In Chapter 2, the grazing flow in the vicinity of the discontinuous boundary will be presented for discontinuous dynamic systems. The accessible and inaccessible subdomains in phase space will be introduced. On the accessible domain, the corresponding, uniquely-continuous, dynamic systems will be defined. The oriented boundary sets and singular sets will be defined from the flow passability on separation boundaries. The local singularity and tangency of a flow on a specified separation boundary will be investigated. The necessary and sufficient conditions for such a local singularity and tangency will be presented. The grazing flows in piecewise linear systems and friction-induced oscillators will be investigated, and the grazing conditions of the flows will be determined.

In Chapter 3, based on the flow nonpassability on a specific separation boundary, the sliding dynamics on the separation boundary will be discussed through the set-valued vector field theory. From the vector fields in the neighborhood of the separation boundary, the passability of the flow from the one domain into another one will be further discussed. The switching bifurcations from the passable boundary to the nonpassable boundary will be discussed. The sliding flow fragmentation mechanism on a specific separation boundary will be presented. The normal vector product function will be introduced to determine the switching bifurcation and sliding fragmentation. The separation boundary formation and properties will also be discussed. The friction-induced oscillator will be presented as an application example to demonstrate sliding flows on the separation boundary.

In Chapter 4, the transversal tangential singularity and bouncing flows in the vicinity of the discontinuous boundary will be discussed. The basic concepts for such transversal tangential singularity will be introduced. The corresponding necessary and sufficient conditions for the transverse, tangential flows on the passable separation boundaries are discussed. The bouncing flows on the boundary will be discussed also. A simple discontinuous dynamical system is presented herein for helping us understand such concepts.

In Chapter 5, the singular sets on the separation boundary will be introduced. In the vicinity of such singular sets, the hyperbolicity and parabolicity of flows will be presented. The real and imaginary flows in nonsmooth dynamical system will be introduced for the onset and vanishing of the sink and source flows. The imaginary flows are defined on the other domains rather than its original domain. The new concept of the imaginary flows will help us to understand the mechanism of sink and source flows in the vicinity of the boundary.

In Chapter 6, discontinuous dynamical systems with flow barriers on the boundary will be discussed for the flow passability from an accessible domain to another neighbored accessible domain. Because the flow barriers exist on the separation boundary, the singularities on the separation boundary will also be influenced by the flow barriers. Therefore, the corresponding necessary and sufficient conditions for the switching bifurcation and sliding fragmentation of flows on the separation will be developed. A periodically forced friction model will be considered for illustration of a flow into the flow barrier with a force jumping.

In Chapter 7, a suitable transport law will be introduced for the flow continuity from an accessible domain to another accessible one in discontinuous dynamical systems. The transport law is as a specific map. Once the specific transport law between the two accessible domains is determined or enforced, the global flow in all possible, accessible domains can continue. Because of the flow connectivity with transport laws, the mapping dynamics of the global flow in all possible, accessible domains will be discussed. The mapping structure will be introduced for the complicated periodic flow. A linear dynamical system with impacting will be presented for demonstrating the methodology to determine the global periodic flow in discontinuous dynamical systems.

Chapter 8 will focus on the symmetry of global flows in discontinuous dynamical systems through mapping techniques. The grazing mapping clusters will be introduced first. Based on the grazing mapping clusters, the mapping structures are used to discuss the flow symmetry of periodic and chaotic motions. The initial and final, switching manifolds are introduced for the strange attractor fragmentation phenomena of chaotic motion in discontinuous dynamical systems. The mechanism of such strange attractor fragmentation will be briefly discussed. The fragmentized strange attractors in discontinuous dynamical systems will be presented through a piecewise linear system.

Chapter 2

Flow Passability and Tangential Flows

In this chapter, the singularity in the vicinity of the discontinuous boundary will be presented. The accessible and inaccessible subdomains will be introduced for development of a theory of nonsmooth dynamic systems on connectable and accessible subdomains. On the accessible domain, the corresponding dynamic systems are introduced. The oriented boundary sets and singular sets caused by the separation boundary will be discussed. The local singularity and tangency of a flow on the separation boundary will be investigated. The necessary and sufficient conditions for such a local singularity and tangency will be presented. The grazing flows in piecewise linear systems and friction-induced oscillators will be investigated, and the grazing conditions of the flows will be determined.

2.1. Domain accessibility

Before development of a general theory for nonsmooth dynamical systems on a universal domain $\mho \subset \Re^n$ in phase space, the subdomains Ω_i $(i = 1, 2, \ldots)$ of the domain $\mho$ are introduced, and the dynamics on the subdomains are defined differently.

DEFINITION 2.1. A subdomain in the universal domain $\mho$ is termed the *accessible* subdomain on which a specific, continuous dynamical system can be defined.

DEFINITION 2.2. A subdomain in a universal domain $\mho$ is termed the *inaccessible* subdomain on which no dynamical system can be defined.

Since the dynamical system can be defined differently on each accessible subdomain, the dynamical behaviors of the system in those accessible subdomains Ω_i can be different from each other in the sense of Newton's mechanics. These different behaviors cause the complexity of motion in the universal domain $\mho$. Owing to the accessible and inaccessible subdomains, the universal domain $\mho$ is classified into the connectable and separable ones. The connectable domain is defined as:

DEFINITION 2.3. A domain $\mho$ in phase space is termed *the connectable domain* if all the accessible subdomains of the universal domain can be connected without any inaccessible subdomain.

Similarly, a definition of the separable domain is:

DEFINITION 2.4. A domain is termed the *separable domain* if the accessible subdomains in the universal domain are separated by inaccessible domains.

The boundary between two adjacent, accessible subdomains is a bridge of dynamical behaviors in two domains for motion continuity. For the connectable domain, it is bounded by the universal boundary surface $S \subset \Re^r$ $(r \leqslant n-1)$, and each subdomain is bounded by the subdomain boundary surface $S_{ij} \subset \Re^r$ $(i, j \in \{1, 2, \ldots\})$ with or without the partial universal boundary. For instance, consider an n-D connectable domain in phase space, as shown in Fig. 2.1(a) through an n_1-dimensional, subvector $\mathbf{x}_{n_1}$ and an $(n-n_1)$-dimensional, subvector $\mathbf{x}_{n-n_1}$. The shaded area Ω_i is a specific subdomain, and other subdomains are white. The dark, solid curve represents the original boundary of the domain $\mho$. In the separable domain, there is at least an inaccessible subdomain to separate the accessible subdomains. The union of inaccessible subdomains is also called the "sea". The sea is the complement of the accessible subdomains to the universal (original) domain $\mho$. That is determined by $\Omega_0 = \mho \setminus \bigcup_i \Omega_i$. The accessible subdomains in the domain $\mho$ are also called the "islands". For illustration of such a definition, an n-D separable domain is shown in Fig. 2.1(b). The dashed surface is the boundary of the universal domain, and the gray area is the sea. The white regions are the accessible domains (or islands). The diagonal line shaded region represents a specific accessible subdomain (island). From one island to another, the transport is needed for motion continuity. The transport laws will be discussed in Chapter 7. For an accessible domain, grazing flows in discontinuous systems will be presented in this chapter.

2.2. Discontinuous dynamic systems

To demonstrate the basic concepts of nonsmooth dynamical system theory, the development of the theory in this chapter is restricted to an n-dimensional, nonsmooth dynamical system. Consider a dynamic system consisting of N subdynamic systems in a universal domain $\mho \subset \Re^n$. The universal domain is divided into N accessible subdomains Ω_i, and the union of all the accessible subdomains $\bigcup_{i=1}^{N} \Omega_i$ and the universal domain $\mho = \bigcup_{i=1}^{N} \Omega_i \cup \Xi$, as shown in Fig. 2.1 through an n_1-dimensional, subvector $\mathbf{x}_{n_1}$ and an $(n-n_1)$-dimensional, subvector $\mathbf{x}_{n-n_1}$. Ω_0 is the union of the inaccessible domains. For the connectable domain

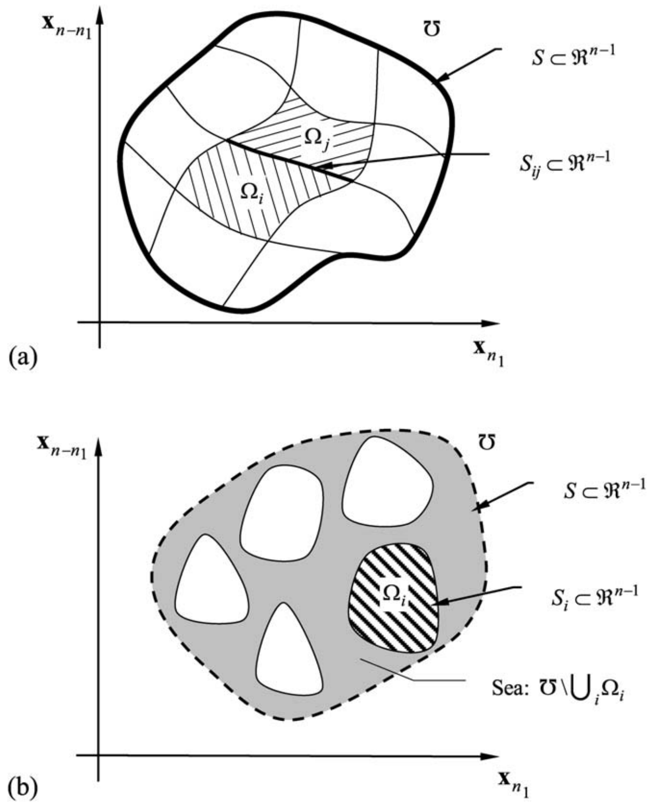


Figure 2.1. Phase space: (a) connectable and (b) separable domains.

in Fig. 2.1(a), $\Omega_0 = \varnothing$. In Fig. 2.1(b), the union of the inaccessible subdomains is the sea, $\Omega_0 = \mho \setminus \bigcup_{i=1}^{m} \Omega_i$ is the complement of the union of the accessible subdomain. On the ith open subdomain Ω_i, there is a C^r-continuous system ($r \geqslant 1$) in the form of

$$\dot{\mathbf{x}} \equiv \mathbf{F}^{(i)}(\mathbf{x}, t, \boldsymbol{\mu}_i) \in \Re^n, \quad \mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \Omega_i. \tag{2.1}$$

The time is t and $\dot{\mathbf{x}} = \mathrm{d}\mathbf{x}/\mathrm{d}t$. In an accessible subdomain Ω_i, the vector field $\mathbf{F}^{(i)}(\mathbf{x}, t, \boldsymbol{\mu}_i)$ with parameter vectors $\boldsymbol{\mu}_i = (\mu_i^{(1)}, \mu_i^{(2)}, \ldots, \mu_i^{(l)})^{\mathrm{T}} \in \Re^l$ is C^r-continuous ($r \geqslant 1$) in $\mathbf{x}$ and for all time t; and the continuous flow in Eq. (2.1) $\mathbf{x}^{(i)}(t) = \boldsymbol{\Phi}^{(i)}(\mathbf{x}^{(i)}(t_0), t, \boldsymbol{\mu}_i)$ with $\mathbf{x}^{(i)}(t_0) = \boldsymbol{\Phi}^{(i)}(\mathbf{x}^{(i)}(t_0), t_0, \boldsymbol{\mu}_i)$ is C^{r+1}-continuous for time t.

The nonsmooth dynamic theory developed in this paper holds for the following conditions:

(A1) The switching between two adjacent subsystems possesses time-continuity.

(A2) For an unbounded, accessible subdomain Ω_i, there is a bounded domain $D_i \subset \Omega_i$ and the corresponding vector field and its flow are bounded, i.e.,

$$\begin{aligned} &\left\|\mathbf{F}^{(i)}\right\| \leqslant K_1 \text{ (const)} \quad \text{and} \\ &\left\|\mathbf{\Phi}^{(i)}\right\| \leqslant K_2 \text{ (const)} \quad \text{on } D_i \text{ for } t \in [0, \infty). \end{aligned} \tag{2.2}$$

(A3) For a bounded, accessible domain Ω_i, there is a bounded domain $D_i \subset \Omega_i$ and the corresponding vector field is bounded, but the flow may be unbounded, i.e.,

$$\left\|\mathbf{F}^{(i)}\right\| \leqslant K_1 \text{ (const)} \quad \text{and} \quad \left\|\mathbf{\Phi}^{(i)}\right\| < \infty \quad \text{on } D_i \text{ for } t \in [0, \infty). \tag{2.3}$$

2.3. Oriented boundary and singular sets

Since dynamical systems on the different accessible subdomains are distinguishing, the relation between flows in the two subdomains should be developed herein for flow continuity. For a subdomain Ω_i, there are k_i-segment boundaries ($k_i \leqslant N-1$). Consider a boundary set of any two subdomains, formed by the intersection of the closed subdomains, i.e., $\partial\Omega_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j$ ($i, j \in \{1, 2, \ldots, N\}$, $j \neq i$), as shown in Fig. 2.2.

DEFINITION 2.5. The boundary in the n-D phase space is defined as

$$\begin{aligned} S_{ij} &\equiv \partial\Omega_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j \\ &= \left\{\mathbf{x} \mid \varphi_{ij}(\mathbf{x}, t) = 0 \text{ where } \varphi_{ij} \text{ is } C^r\text{-continuous } (r \geqslant 1)\right\} \subset \Re^{n-1}. \end{aligned} \tag{2.4}$$

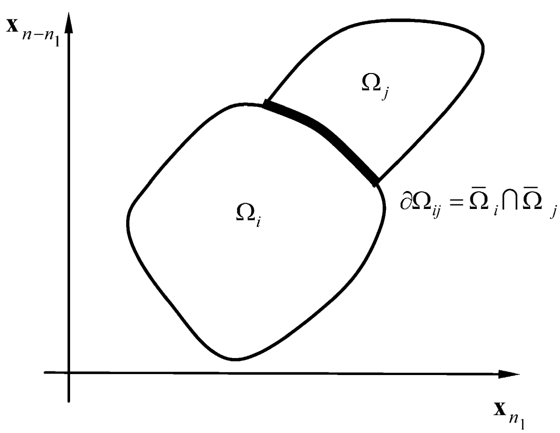


Figure 2.2. Subdomains Ω_i and Ω_j, the corresponding boundary $\partial\Omega_{ij}$.

DEFINITION 2.6. The two subdomains Ω_i and Ω_j are *disjoint* if the boundary $\partial\Omega_{ij}$ is an empty set (i.e., $\partial\Omega_{ij} = \varnothing$).

The boundary values $\mathbf{x}^{(\alpha)} = (x_1^{(\alpha)}, x_2^{(\alpha)}, \ldots, x_n^{(\alpha)})^{\mathrm{T}}$, $\alpha \in \{i, j\}$, pertain to the open domains Ω_i and Ω_j, respectively. Based on the boundary definition, we have $\partial\Omega_{ij} = \partial\Omega_{ji}$.

The boundary $\partial\Omega_{ij}$ with $\varphi_{ij}(\mathbf{x}, t) = 0$ can be determined by

$$\dot{\mathbf{x}}^{(0)} = \mathbf{F}^{(0)}\big(\mathbf{x}^{(0)}, t\big) \tag{2.5}$$

where $\mathbf{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})^{\mathrm{T}}$.

DEFINITION 2.7. If the intersection of the three or more subdomains,

$$\Gamma_{i_1 i_2 \cdots i_k} \equiv \bigcap_{i=i_1}^{i_k} \overline{\Omega}_i \subset \Re^r \quad (r = 0, 1, \ldots, N-2) \tag{2.6}$$

where $i_k \in \{1, 2, \ldots, n_2\}$ and $k \geqslant 3$, is nonempty, the subdomain intersection is termed the *singular* set.

The boundary functions relative to the singular sets are C^0-continuous. For $r = 0$, the singular sets will be singular points, which are also termed *the corner points or vertex*. For $r = 1$, the singular sets will be lines, which are termed the singular edges to the $(n-1)$-dimensional discontinuous boundary. For $r \in \{2, 3, \ldots, n-2\}$, the singular sets are the r-dimensional surfaces to the $(n-1)$-dimensional discontinuous boundary. In Fig. 2.3, the singular set for three closed domains $\{\overline{\Omega}_i, \overline{\Omega}_j, \overline{\Omega}_k\}$ is sketched. The circular symbols represent intersection sets.

The largest solid circular symbol stands for the singular set Γ_{ijk}. The corresponding discontinuous boundaries relative to the singular set are labeled by $\partial\Omega_{ij}$, $\partial\Omega_{jk}$ and $\partial\Omega_{ik}$. The singular set possesses the hyperbolic or parabolic behavior depending on the properties of the discontinuous boundary set, which can be referred to Luo (2005a). The detailed discussion is given in Chapter 5.

DEFINITION 2.8. For a discontinuous dynamical system in Eq. (2.1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$, suppose $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. The nonempty boundary set $\partial\Omega_{ij}$ to a flow $\mathbf{x}^{(\alpha)}(t)$ $(\alpha \in \{i, j\})$ is *semi-passable* from the domain Ω_i to Ω_j (expressed by $\overrightarrow{\partial\Omega_{ij}}$) if the flow $\mathbf{x}^{(\alpha)}(t)$ possesses the following properties:

$$\begin{aligned} \text{either} \quad & \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \big[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})\big] > 0 \quad \text{and} \\ & \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \big[\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\big] > 0 \\ & \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j, \end{aligned} \tag{2.7a}$$

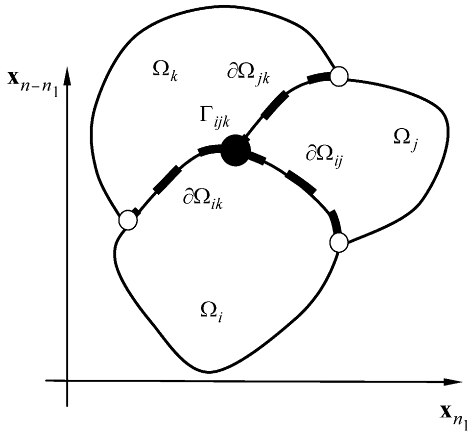


Figure 2.3. A singular set for the intersection of three domains $\{\overline{\Omega}_i, \overline{\Omega}_j, \overline{\Omega}_k\}$. The circular circles represent intersection sets. The largest solid circular symbol stands for the singular set Γ_{ijk}. The corresponding discontinuous boundaries are marked by $\partial\Omega_{ij}$, $\partial\Omega_{jk}$ and $\partial\Omega_{ik}$.

$$\begin{aligned}\text{or}\quad & \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})\right] < 0 \quad \text{and} \\ & \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] < 0 \\ & \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i, \end{aligned} \tag{2.7b}$$

where the normal vector of the boundary $\partial\Omega_{ij}$ is

$$\mathbf{n}_{\partial\Omega_{ij}} = \nabla\varphi_{ij} = \left(\frac{\partial\varphi_{ij}}{\partial x_1}, \frac{\partial\varphi_{ij}}{\partial x_2}, \ldots, \frac{\partial\varphi_{ij}}{\partial x_n}\right)^{\mathrm{T}}_{(\mathbf{x}_m)}. \tag{2.8}$$

The notation $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$ means that the normal vector of the boundary $\partial\Omega_{ij}$ points to the domain Ω_j, and $\mathbf{x}_m \equiv \mathbf{x}^{(0)}(t_{m\pm})$. The notations $t_{m\pm\varepsilon} = t_m \pm \varepsilon$ and $t_{m\pm} = t_m \pm 0$ are used. To interpret the geometrical concept of the semi-passable boundary sets, let us consider a flow in Eq. (2.1) from the domain Ω_i into the domain Ω_j through the boundary $\partial\Omega_{ij}$. At a time t_m, the flow arrives to the boundary $\partial\Omega_{ij}$, and there is a small neighborhood $(t_{m-\varepsilon}, t_{m+\varepsilon})$ of the time t_m, which is arbitrarily selected. Here $t_{m\pm\varepsilon} \triangleq t_m \pm \varepsilon$. As $\varepsilon \to 0$, the time increment $\Delta t \equiv \varepsilon \to 0$. Before the flow reaches to the boundary, the point $\mathbf{x}^{(i)}(t_{m-\varepsilon})$ lies in the domain Ω_i. The point $\mathbf{x}^{(i)}_m$ is for the flow on the boundary. After the flow passes through the boundary, $\mathbf{x}^{(i)}(t_{m+\varepsilon})$ is a point in the neighborhood of the discontinuous boundary on the side of the domain Ω_j. The input and output flow vectors are $\mathbf{x}^{(i)}(t_m) - \mathbf{x}^{(i)}(t_{m-\varepsilon})$ and $\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(j)}(t_m)$, respectively. Whether the flow passed through the boundary or not is dependent on the properties of both

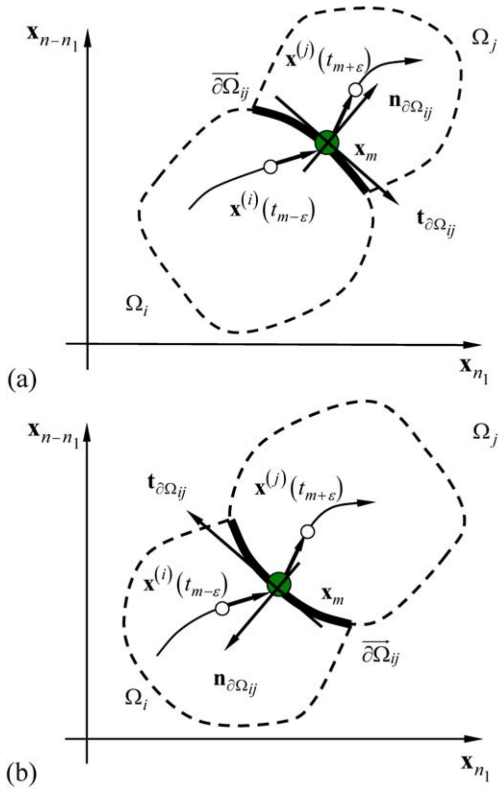


Figure 2.4. Semi-passable boundary set $\overrightarrow{\partial\Omega}_{ij}$ for the flow passing boundary from the domain Ω_i to Ω_j: (a) $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$, and (b) $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i$. $\mathbf{x}^{(i)}(t_{m-\varepsilon})$, $\mathbf{x}^{(j)}(t_{m+\varepsilon})$ and $\mathbf{x}_m$ are three points in the domains Ω_i and Ω_j and on the boundary $\partial\Omega_{ij}$, respectively. Two vectors $\mathbf{n}_{\partial\Omega_{ij}}$ and $\mathbf{t}_{\partial\Omega_{ij}}$ are the normal and tangential vectors of $\partial\Omega_{ij}$.

input and outflow vectors in the neighborhood of the boundary. The process of the flow passing through the boundaries $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$ and $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i$ from the domain Ω_i to Ω_j is shown in Fig. 2.4. Two vectors $\mathbf{n}_{\partial\Omega_{ij}}$ and $\mathbf{t}_{\partial\Omega_{ij}}$ are the normal and tangential vectors of the boundary curve $\partial\Omega_{ij}$ determined by $\varphi_{ij}(\mathbf{x}, t) = 0$ with Eq. (2.5). When an input flow $\mathbf{x}^{(i)}(t)$ in the domain Ω_i arrives to semi-passable boundary $\overrightarrow{\partial\Omega}_{ij}$, the flow can be tangential to, bouncing on semi-passable boundary $\overrightarrow{\partial\Omega}_{ij}$, or the flow can be tangential to, or bouncing on, or passing through the semi-passable boundary $\overrightarrow{\partial\Omega}_{ij}$. However, once a flow $\mathbf{x}^{(j)}(t)$ in the domain Ω_j arrives to the semi-passable boundary $\overrightarrow{\partial\Omega}_{ij}$, the flow cannot pass through the boundary, and either the tangential or bouncing flow $\mathbf{x}^{(j)}(t)$ at the semi-passable boundary $\overrightarrow{\partial\Omega}_{ij}$ exists. The tangential (or grazing) flows will be discussed in this chapter. In the following discussion, no control and transport laws are defined on the semi-passable boundary. The direction of $\mathbf{t}_{\partial\Omega_{ij}} \times \mathbf{n}_{\partial\Omega_{ij}}$ is the positive direction of the coordinate by the right-hand rule.

THEOREM 2.1. *For a discontinuous dynamical system in Eq. (2.1),* $\mathbf{x}(t_m) = \mathbf{x}_m \in \partial\Omega_{ij}$ *at* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$. *Suppose* $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$, *both flows* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *are* $C^r_{[t_{m-\varepsilon},t_m)}$*- and* $C^r_{(t_m,t_{m+\varepsilon}]}$*-continuous* ($r \geqslant 2$) *for time* t, *respectively, and* $\|\mathrm{d}^r\mathbf{x}^{(\alpha)}/\mathrm{d}t^r\| < \infty$ ($\alpha \in \{i, j\}$). *The nonempty boundary set* $\partial\Omega_{ij}$ *for the flows* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *is semi-passable from the domain* Ω_i *to* Ω_j *iff*

$$\begin{aligned} &\textit{either} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(i)}(t_{m-}) > 0 \quad \textit{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(j)}(t_{m+}) > 0 \\ &\qquad\qquad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j, \\ &\textit{or} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(i)}(t_{m-}) < 0 \quad \textit{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(j)}(t_{m+}) < 0 \\ &\qquad\qquad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{aligned} \tag{2.9}$$

PROOF. For a point $\mathbf{x}_m \in \partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$, suppose $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$ and, both $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^r_{[t_{m-\varepsilon},t_m)}$- and $C^r_{(t_m,t_{m+\varepsilon}]}$-continuous ($r \geqslant 2$) for time t, respectively and $\|\ddot{\mathbf{x}}^{(\alpha)}(t)\| < \infty$ ($\alpha \in \{i, j\}$) for $0 < \varepsilon \ll 1$. Consider $a \in [t_{m-\varepsilon}, t_{m-})$ and $b \in (t_{m-}, t_{m+\varepsilon}]$. Application of the Taylor series expansion of $\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon})$ with $t_{m\pm\varepsilon} = t_m \pm \varepsilon$ ($\alpha \in \{i, j\}$) to $\mathbf{x}^{(\alpha)}(a)$ and $\mathbf{x}^{(\alpha)}(b)$ gives

$$\begin{aligned} \mathbf{x}^{(i)}(t_{m-\varepsilon}) &\equiv \mathbf{x}^{(i)}(t_{m-} - \varepsilon) \\ &= \mathbf{x}^{(i)}(a) + \dot{\mathbf{x}}^{(i)}(a)(t_{m-} - \varepsilon - a) + o\big((t_{m-} - \varepsilon - a)\big), \end{aligned}$$

$$\begin{aligned} \mathbf{x}^{(j)}(t_{m+\varepsilon}) &\equiv \mathbf{x}^{(j)}(t_{m+} + \varepsilon) \\ &= \mathbf{x}^{(j)}(b) + \dot{\mathbf{x}}^{(j)}(t_{m+} + \varepsilon - b)\varepsilon + o\big((t_{m+} + \varepsilon - b)\big). \end{aligned}$$

Let $a \to t_{m-}$ and $b \to t_{m+}$, the limits of the foregoing equations lead to

$$\begin{aligned} \mathbf{x}^{(i)}(t_{m-\varepsilon}) &\equiv \mathbf{x}^{(i)}(t_{m-} - \varepsilon) = \mathbf{x}^{(i)}(t_{m-}) - \dot{\mathbf{x}}^{(i)}(t_{m-})\varepsilon + o(\varepsilon), \\ \mathbf{x}^{(j)}(t_{m+\varepsilon}) &\equiv \mathbf{x}^{(j)}(t_{m+} + \varepsilon) = \mathbf{x}^{(j)}(t_{m+}) + \dot{\mathbf{x}}^{(j)}(t_{m+})\varepsilon + o(\varepsilon). \end{aligned}$$

Because of $0 < \varepsilon \ll 1$, the ε^2 and higher order terms of the foregoing equations can be ignored. Therefore, with the first part of Eq. (2.9), the following relations exist,

$$\begin{aligned} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathbf{x}^{(i)}(t_{m-}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})\big] &= \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(i)}(t_{m-})\varepsilon > 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(j)}(t_{m+})\big] &= \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(i)}(t_{m+})\varepsilon > 0. \end{aligned}$$

Similarly,

$$\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \left[\mathbf{x}^{(0)}(t_{m-}) - \mathbf{x}^{(0)}(t_{m-\varepsilon})\right] = \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(0)}(t_{m-})\varepsilon = 0,$$
$$\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \left[\mathbf{x}^{(0)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+})\right] = \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}(t_{m+})\varepsilon = 0.$$

Therefore,

$$\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})\right] = \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(i)}(t_{m-})\varepsilon > 0,$$
$$\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \left[\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] = \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(j)}(t_{m+})\varepsilon > 0.$$

From Definition 2.8, the boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$ is semi-passable under the conditions in the first part of Eq. (2.9). In a similar manner, the boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i$ is semi-passable under the conditions in the second part of Eq. (2.9). □

THEOREM 2.2. *For a discontinuous dynamical system in Eq.* (2.1), $\mathbf{x}(t_m) = \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$. *Suppose* $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$, *both* $\mathbf{F}^{(i)}(t)$ *and* $\mathbf{F}^{(j)}(t)$ *are* $C^r_{[t_{m-\varepsilon},t_m)}$*- and* $C^r_{(t_m,t_{m+\varepsilon}]}$*-continuous* ($r \geqslant 1$) *for time* t, *respectively, and* $\|\mathrm{d}^{r+1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r+1}\| < \infty$ ($\alpha \in \{i, j\}$). *The nonempty boundary set* $\partial\Omega_{ij}$ *for the flows* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *is semi-passable from the domain* Ω_i *to* Ω_j *iff*

$$\begin{aligned} &\textit{either} \quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}^{(i)}(t_{m-}) > 0 \quad \textit{and} \quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}^{(j)}(t_{m+}) > 0 \\ &\qquad\qquad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j, \\ &\textit{or} \qquad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}^{(i)}(t_{m-}) < 0 \quad \textit{and} \quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}^{(j)}(t_{m+}) < 0 \\ &\qquad\qquad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i \end{aligned} \tag{2.10}$$

where $\mathbf{F}^{(i)}(t_{m-}) = \mathbf{F}^{(i)}(\mathbf{x}, t_{m-}, \boldsymbol{\mu}_i)$ *and* $\mathbf{F}^{(j)}(t_{m+}) = \mathbf{F}^{(j)}(\mathbf{x}, t_{m+}, \boldsymbol{\mu}_j)$.

PROOF. For a point $\mathbf{x}_m \in \partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$, the relationship $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$ exists. With Eq. (2.1), the first part of Eq. (2.10) gives

$$\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(i)}(t_{m-}) = \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}^{(i)}(t_{m-}) > 0 \quad \text{and}$$
$$\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(j)}(t_{m+}) = \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}^{(j)}(t_{m+}) > 0.$$

From Theorem 2.1 and Definition 2.8, the boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$ is semi-passable. In a similar fashion, the boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i$ is semi-passable under the condition in the second part of Eq. (2.10). □

DEFINITION 2.9. For a discontinuous dynamical system in Eq. (2.1), $\mathbf{x}(t_m) = \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_m)$ and suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}_m$ ($\alpha \in \{i, j\}$). The nonempty boundary set $\partial\Omega_{ij}$

for flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ is *the nonpassable boundary of the first kind*, $\widetilde{\partial\Omega}_{ij}$ (or termed *a sink boundary* between the subdomains Ω_i and Ω_j) if the flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ in the neighborhood of the boundary $\partial\Omega_{ij}$ possess the following properties:

$$\big\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})\big]\big\}\big\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(j)}(t_{m-\varepsilon})\big]\big\} < 0. \tag{2.11}$$

DEFINITION 2.10. For a discontinuous dynamical system in Eq. (2.1), $\mathbf{x}(t_m) = \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there is a time interval $(t_m, t_{m+\varepsilon}]$ and suppose $\mathbf{x}^{(\alpha)}(t_{m+}) = \mathbf{x}_m$ ($\alpha \in \{i, j\}$). The nonempty boundary set $\partial\Omega_{ij}$ for the flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ is *the nonpassable boundary of the second kind* $\widehat{\partial\Omega}_{ij}$ (or termed *a source boundary* between the subdomains Ω_i and Ω_j) if the flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ in the neighborhood of the boundary $\partial\Omega_{ij}$ possess the following properties:

$$\big\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathbf{x}^{(i)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\big]\big\}\big\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\big]\big\} < 0. \tag{2.12}$$

The above two concepts for the sink and source boundaries between the two subdomains Ω_i and Ω_j are illustrated in Figs. 2.5(a) and (b). The flows in the neighborhood of the boundaries are depicted. When a flow $\mathbf{x}^{(\alpha)}(t)$ ($\alpha \in \{i, j\}$) in the domain Ω_α arrives to the nonpassable boundary of the first kind $\widetilde{\partial\Omega}_{ij}$, the flow can be tangential to or sliding on the nonpassable boundary $\widetilde{\partial\Omega}_{ij}$. For the nonpassable boundary of the second kind $\widehat{\partial\Omega}_{ij}$, a flow $\mathbf{x}^{(\alpha)}(t)(\alpha \in \{i, j\})$ in the domain Ω_α can be tangential to or bouncing on the nonpassable boundary $\widehat{\partial\Omega}_{ij}$. In this chapter, only the tangential flows on the nonpassable boundary will be discussed, which are similar to the one on the semi-passable boundary.

THEOREM 2.3. *For a discontinuous dynamical system in Eq.* (2.1), $\mathbf{x}(t_m) = \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there is a time interval* $[t_{m-\varepsilon}, t_m)$ *and suppose* $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}_m$ ($\alpha \in \{i, j\}$). *The flow* $\mathbf{x}^{(\alpha)}(t)$ *is* $C^r_{[t_{m-\varepsilon}, t_m)}$*-continuous* ($r \geqslant 2$) *for time* t, *and* $\|\mathrm{d}^r\mathbf{x}^{(\alpha)}/\mathrm{d}t^r\| < \infty$. *The nonempty boundary set* $\partial\Omega_{ij}$ *for the flows* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *is a nonpassable boundary of the first kind iff*

$$\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(i)}(t_{m-})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(j)}(t_{m-})\big] < 0. \tag{2.13}$$

PROOF. The theorem can be proved following the procedure of the proof of Theorem 2.1. □

THEOREM 2.4. *For a discontinuous dynamical system in Eq.* (2.1), $\mathbf{x}(t_m) = \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there is a time interval* $[t_{m-\varepsilon}, t_m)$

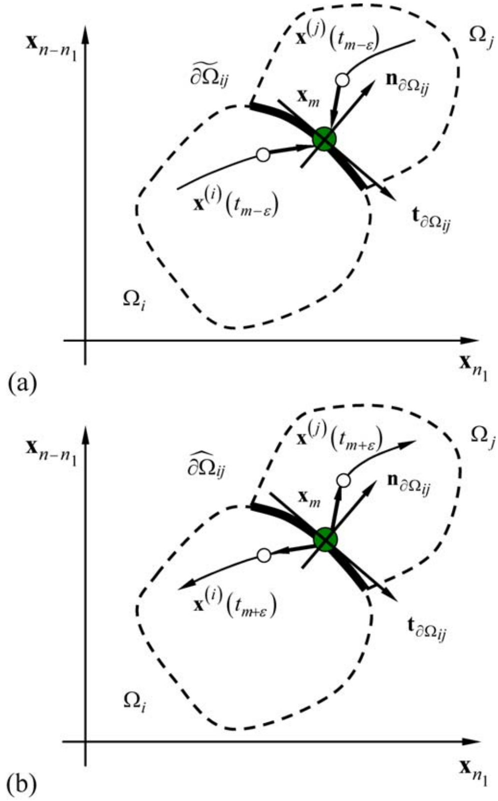


Figure 2.5. Nonpassable boundary set $\overline{\partial\Omega}_{ij} = \widetilde{\partial\Omega}_{ij} \cup \widehat{\partial\Omega}_{ij}$: (a) the sink boundary (or the nonpassable boundary of the first kind, $\widetilde{\partial\Omega}_{ij}$), (b) the source boundary (or the nonpassable boundary of the second kind, $\widehat{\partial\Omega}_{ij}$). $\mathbf{x}_m \equiv (\mathbf{x}_{n_1}(t_m), \mathbf{x}_{n-n_1}(t_m))^{\mathrm{T}}$, $\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon}) \equiv (\mathbf{x}^{(\alpha)}_{n_1}(t_{m\pm\varepsilon}), \mathbf{x}^{(\alpha)}_{n-n_1}(t_{m\pm\varepsilon}))^{\mathrm{T}}$ and $\alpha \in \{i, j\}$ where $t_{m\pm\varepsilon} = t_m \pm \varepsilon$ for an arbitrary small $\varepsilon > 0$.

and suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}_m$ ($\alpha \in \{i, j\}$). *The vector field* $\mathbf{F}^{(\alpha)}(t)$ *is* $C^r_{[t_{m-\varepsilon}, t_m)}$*-continuous* ($r \geqslant 1$), *and* $\|\mathrm{d}^{r+1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r+1}\| < \infty$. *The nonempty boundary set* $\partial\Omega_{ij}$ *for the flows* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *is a nonpassable boundary of the first kind iff*

$$\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-})\right]\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m-})\right] < 0 \tag{2.14}$$

where $\mathbf{F}^{(\alpha)}(t_{m-}) \triangleq \mathbf{F}^{(\alpha)}(\mathbf{x}, t_{m-}, \boldsymbol{\mu}_\alpha)$ ($\alpha \in \{i, j\}$).

PROOF. The theorem can be proved following the procedure of the proof of Theorem 2.2. □

THEOREM 2.5. *For a discontinuous dynamical system in Eq.* (2.1), $\mathbf{x}(t_m) = \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there is a time interval* $(t_m, t_{m+\varepsilon}]$ *and suppose* $\mathbf{x}^{(\alpha)}(t_{m+}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$. $\mathbf{x}^{(\alpha)}(t)$ *is* $C^r_{(t_m, t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 2)$ *for time* t, *and* $\|\mathrm{d}^r\mathbf{x}^{(\alpha)}/\mathrm{d}t^r\| < \infty$. *The nonempty boundary set* $\partial\Omega_{ij}$ *for the flows* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *is a nonpassable boundary of the second kind iff*

$$\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(i)}(t_{m+})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(j)}(t_{m+})\big] < 0. \tag{2.15}$$

PROOF. The theorem can be proved following the procedure of the proof of Theorem 2.1. □

THEOREM 2.6. *For a discontinuous dynamical system in Eq.* (2.1), $\mathbf{x}(t_m) = \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there is a time interval* $(t_m, t_{m+\varepsilon}]$ *and suppose* $\mathbf{x}^{(\alpha)}(t_{m+}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$. *The vector field* $\mathbf{F}^{(\alpha)}(t)$ *is* $C^r_{[t_{m-\varepsilon}, t_m)}$*-continuous* $(r \geqslant 1)$, *and* $\|\mathrm{d}^{r+1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r+1}\| < \infty$. *The nonempty boundary set* $\partial\Omega_{ij}$ *for the flows* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *is a nonpassable boundary of the second kind iff*

$$\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m+})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+})\big] < 0 \tag{2.16}$$

where $\mathbf{F}^{(\alpha)}(t_{m+}) \triangleq \mathbf{F}^{(\alpha)}(\mathbf{x}, t_{m+}, \boldsymbol{\mu}_\alpha)$ $(\alpha \in \{i, j\})$.

PROOF. The theorem can be proved following the procedure of the proof of Theorem 2.2. □

DEFINITION 2.11. The nonempty boundary set $\partial\Omega_{ij}$ for the flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ is passable $(\overleftrightarrow{\partial\Omega}_{ij})$ only if it possesses not only semi-passable boundary $\overrightarrow{\partial\Omega}_{ij}$ from the domain Ω_i to Ω_j but $\overleftarrow{\partial\Omega}_{ij}$ from the domain Ω_j to Ω_i.

This definition indicates that the C^0-flow on the boundary set is invertible. The gradients of the flow on both sides of the separation boundary are different in the nonsmooth dynamical systems. If the flow is C^1-continuous on the boundary without effects of sliding motion, the boundary set becomes a trivial boundary set, and the two subdynamical systems become a smooth dynamical system. For illustration of the passable boundary set, the flows passing through the boundary $\partial\Omega_{ij}$ from Ω_i to Ω_j and from Ω_j to Ω_i are presented in Fig. 2.6. The dashed curves are other boundaries for the domains Ω_i and Ω_j. The thicker solid curve represents the boundary $\partial\Omega_{ij}$. The thinner solid curves with arrows are the flow of Eq. (2.1) in the two domains.

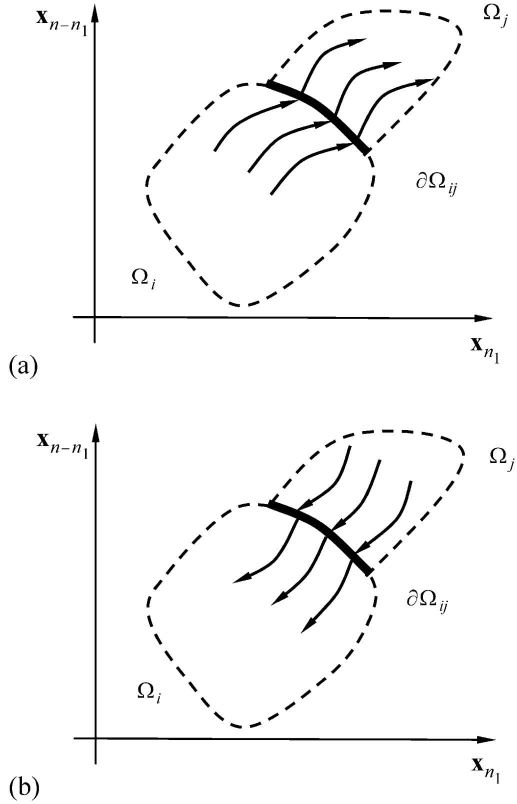


Figure 2.6. Flow passing through the boundary $\partial\Omega_{ij}$: (a) from Ω_i to Ω_j, and (b) from Ω_j to Ω_i.

2.4. Local singularity and tangential flows

In this section, a local singularity of flow and tangential flows will be discussed. The corresponding necessary and sufficient conditions will be presented.

DEFINITION 2.12. For a discontinuous dynamical system in Eq. (2.1), $\mathbf{x}(t_m) = \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$) and suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(\beta)}(t_{m+})$ ($\alpha, \beta \in \{i, j\}$). The two flows $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^r_{[t_{m-\varepsilon}, t_m)}$- and $C^r_{(t_m, t_{m+\varepsilon}]}$-continuous ($r \geqslant 2$), respectively. A point $\mathbf{x}_m$ is critical on the nonempty boundary set $\partial\Omega_{ij}$ if

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) = 0 \quad \text{and/or} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+}) = 0. \tag{2.17}$$

THEOREM 2.7. *For a discontinuous dynamical system in Eq.* (2.1), $\mathbf{x}(t_m) = \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$*) and suppose* $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(\beta)}(t_{m+})$ $(\alpha, \beta \in \{i, j\})$. *The two vector fields* $\mathbf{F}^{(\alpha)}(t)$ *and* $\mathbf{F}^{(\beta)}(t)$ *are* $C^r_{[t_{m-\varepsilon}, t_m)}$- *and* $C^r_{(t_m, t_{m+\varepsilon}]}$-*continuous* $(r \geqslant 1)$ *for time t, respectively, and* $\|\mathrm{d}^{r+1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r+1}\| < \infty$ $(\alpha \in \{i, j\})$. *The point* $\mathbf{x}_m \in \partial\Omega_{ij}$ *is critical on the nonempty boundary* $\partial\Omega_{ij}$ *iff*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-}) = 0 \quad \textit{and/or} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}(t_{m+}) = 0 \tag{2.18}$$

where $\mathbf{F}^{(\alpha)}(t_{m-}) = \mathbf{F}^{(\alpha)}(\mathbf{x}, t_{m-}, \boldsymbol{\mu}_\alpha)$ *and* $\mathbf{F}^{(\beta)}(t_{m+}) = \mathbf{F}^{(\beta)}(\mathbf{x}, t_{m+}, \boldsymbol{\mu}_\beta)$.

PROOF. Using Eq. (2.1) and Definition 2.12, the Theorem 2.7 can be proved. □

Since the tangential vector of the input and output flows $\mathbf{x}^{(\alpha)}(t_{m-})$ and $\mathbf{x}^{(\alpha)}(t_{m+})$ on the side of the domain Ω_α $(\alpha \in \{i, j\})$ at the boundary $\partial\Omega_{ij}$ is normal to the normal vector of the boundary, it implies that the input flow is tangential to the boundary. The mathematical description is given as follows.

DEFINITION 2.13. For a discontinuous dynamical system in Eq. (2.1), $\mathbf{x}(t_m) = \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$) and suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$. The flow $\mathbf{x}^{(\alpha)}(t)$ is $C^r_{[t_{m-\varepsilon}, t_m)}$- and $C^r_{(t_m, t_{m+\varepsilon}]}$-continuous $(r \geqslant 1)$ for time t. The flow $\mathbf{x}^{(\alpha)}(t)$ in Ω_α is tangential to the boundary $\partial\Omega_{ij}$ if the following two conditions hold:

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m\pm}) = 0 \quad \text{and} \tag{2.19}$$

either

$$\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] > 0, \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] < 0 \\ &\quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta \end{aligned} \tag{2.20}$$

where $\beta \in \{i, j\}$ but $\alpha \neq \beta$, or

$$\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] < 0, \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] > 0 \\ &\quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha. \end{aligned} \tag{2.21}$$

Since $\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{t}_{\partial\Omega_{ij}} = 0$ and $\mathbf{t}_{\partial\Omega_{ij}} = \dot{\mathbf{x}}_m$ on the boundary $\partial\Omega_{ij}$, with Eq. (2.19), we have

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m\pm}) = 0 = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}_m \quad \text{or} \quad \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) = \dot{\mathbf{x}}_m = \dot{\mathbf{x}}^{(\alpha)}(t_{m+}). \tag{2.22}$$

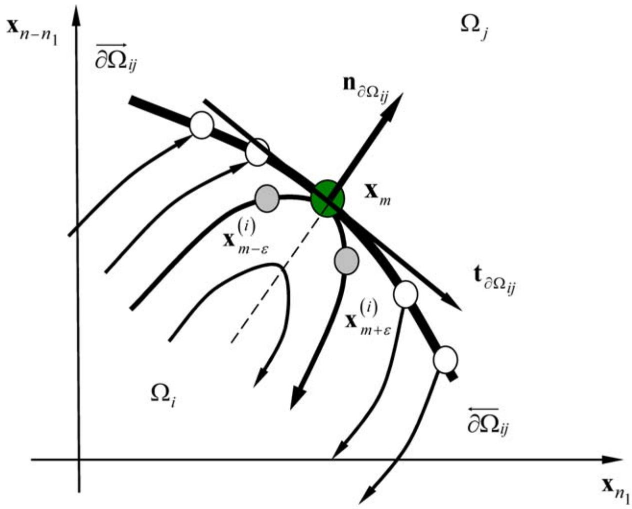


Figure 2.7. A flow in the domain Ω_i tangential to the boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$. The gray-filled symbols represent two points ($\mathbf{x}^{(i)}_{m-\varepsilon}$ and $\mathbf{x}^{(i)}_{m+\varepsilon}$) on the flow before and after the tangency. The tangential point $\mathbf{x}_m$ on the boundary $\partial\Omega_{ij}$ is depicted by a large circular symbol.

The above equation implies that the flow $\mathbf{x}^{(\alpha)}$ on the boundary is at least C^1-continuous. To demonstrate the above definition, consider a flow in the domain Ω_i tangential to the boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$, as shown in Fig. 2.7. The gray-filled symbols represent two points ($\mathbf{x}^{(i)}_{m\pm\varepsilon} = \mathbf{x}^{(i)}(t_m \pm \varepsilon)$) on the flow before and after the tangency. The tangential point $\mathbf{x}_m$ on the boundary $\partial\Omega_{ij}$ is depicted by a large circular symbol. This tangential flow is also termed *the grazing flow*.

THEOREM 2.8. *For a discontinuous dynamical system in Eq.* (2.1), $\mathbf{x}(t_m) = \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals* (*i.e.*, $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$) *and suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$. *The flow* $\mathbf{x}^{(\alpha)}(t)$ *is* $C^r_{[t_{m-\varepsilon}, t_m)}$- *and* $C^r_{(t_m, t_{m+\varepsilon}]}$-*continuous* $(r \geqslant 2)$ *for time t and* $\|\mathrm{d}^r\mathbf{x}^{(\alpha)}/\mathrm{d}t^r\| < \infty$. *The flow* $\mathbf{x}^{(\alpha)}(t)$ *in* Ω_α *is tangential to the boundary* $\partial\Omega_{ij}$ *iff*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m\pm}) = 0, \tag{2.23}$$

$$\left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\dot{\mathbf{x}}^{(0)}(t_{m-\varepsilon}) - \dot{\mathbf{x}}^{(\alpha)}(t_{m-\varepsilon})\right]\right\} \times \left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\dot{\mathbf{x}}^{(\alpha)}(t_{m+\varepsilon}) - \dot{\mathbf{x}}^{(0)}(t_{m+\varepsilon})\right]\right\} < 0. \tag{2.24}$$

PROOF. Since Eq. (2.23) is identical to Eq. (2.19), the first condition in Eq. (2.19) is satisfied:

$$\begin{aligned}\mathbf{x}^{(\alpha)}(t_{m\pm}) &\equiv \mathbf{x}^{(\alpha)}(t_{m\pm} \pm \varepsilon \mp \varepsilon) = \mathbf{x}^{(\alpha)}(t_{m\pm} \pm \varepsilon) \mp \varepsilon\dot{\mathbf{x}}^{(\alpha)}(t_{m\pm} \pm \varepsilon) + o(\varepsilon)\\ &= \mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon}) \mp \varepsilon\dot{\mathbf{x}}^{(\alpha)}(t_{m\pm\varepsilon}) + o(\varepsilon).\end{aligned}$$

For $0 < \varepsilon \ll 1$, the higher order terms in the above equation can be ignored. Therefore,

$$\begin{aligned}&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] = \varepsilon\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-\varepsilon}),\\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})\right] = \varepsilon\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+\varepsilon}).\end{aligned}$$

Similarly,

$$\begin{aligned}&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(0)}(t_{m}) - \mathbf{x}^{(0)}(t_{m-\varepsilon})\right] = \varepsilon\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(0)}(t_{m-\varepsilon}),\\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(0)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m})\right] = \varepsilon\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(0)}(t_{m+\varepsilon}).\end{aligned}$$

From Eq. (2.24), the first case is:

$$\begin{aligned}&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\dot{\mathbf{x}}^{(0)}(t_{m-\varepsilon}) - \dot{\mathbf{x}}^{(\alpha)}(t_{m-\varepsilon})\right] > 0 \quad \text{and}\\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\dot{\mathbf{x}}^{(\alpha)}(t_{m+\varepsilon}) - \dot{\mathbf{x}}^{(0)}(t_{m+\varepsilon})\right] < 0\end{aligned}$$

with which Eq. (2.20) holds for $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta$ ($\beta \neq \alpha$). However, the second case is:

$$\begin{aligned}&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\dot{\mathbf{x}}^{(0)}(t_{m-\varepsilon}) - \dot{\mathbf{x}}^{(\alpha)}(t_{m-\varepsilon})\right] < 0 \quad \text{and}\\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\dot{\mathbf{x}}^{(\alpha)}(t_{m+\varepsilon}) - \dot{\mathbf{x}}^{(0)}(t_{m+\varepsilon})\right] > 0\end{aligned}$$

from which Eq. (2.21) holds for $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha$. Therefore, from Definition 2.13, the flow $\mathbf{x}^{(\alpha)}(t)$ for $t \in T_m$ in Ω_α is tangential to the boundary $\partial\Omega_{ij}$. □

THEOREM 2.9. *For a discontinuous dynamical system in Eq. (2.1),* $\mathbf{x}(t_m) = \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$) *and suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ ($\alpha \in \{i, j\}$). *The vector field* $\mathbf{F}^{(\alpha)}(t)$ *is* $C^r_{[t_{m-\varepsilon},t_m)}$*- and* $C^r_{(t_m,t_{m+\varepsilon}]}$*-continuous* ($r \geqslant 1$) *for time t and* $\|\mathrm{d}^{r+1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r+1}\| < \infty$. *The flow* $\mathbf{x}^{(\alpha)}(t)$ *in* Ω_α *is tangential to the boundary* $\partial\Omega_{ij}$ *iff*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m\pm}) = 0, \tag{2.25}$$

$$\begin{aligned}&\left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{F}^{(\alpha)}(t_{m-\varepsilon}) - \mathbf{F}^{(0)}(t_{m-\varepsilon})\right]\right\}\\ &\quad \times \left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{F}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{F}^{(0)}(t_{m+\varepsilon})\right]\right\} < 0.\end{aligned} \tag{2.26}$$

PROOF. Using Eq. (2.1) and the Theorem 2.8, the Theorem 2.9 can be proved. □

THEOREM 2.10. *For a discontinuous dynamical system in Eq.* (2.1), $\mathbf{x}(t_m) = \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals* (*i.e.*, $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$) *and suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$. *The flow* $\mathbf{x}^{(\alpha)}(t)$ *is* $C^r_{[t_{m-\varepsilon}, t_m)}$- *and* $C^r_{(t_m, t_{m+\varepsilon}]}$-*continuous* $(r \geqslant 3)$ *for time* t *and* $\|\mathrm{d}^r\mathbf{x}^{(\alpha)}/\mathrm{d}t^r\| < \infty$. *The flow* $\mathbf{x}^{(\alpha)}(t)$ *in* Ω_α *is tangential to the boundary* $\partial\Omega_{ij}$ *iff*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m\pm}) = 0 \quad \textit{and} \tag{2.27}$$

$$\begin{aligned} &\textit{either} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\ddot{\mathbf{x}}^{(\alpha)}(t_{m\pm}) - \ddot{\mathbf{x}}^{(0)}(t_m)\big] < 0 \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta \\ &\qquad\qquad (\beta \in \{i, j\}, \beta \neq \alpha), \\ &\textit{or} \qquad\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\ddot{\mathbf{x}}^{(\alpha)}(t_{m\pm}) - \ddot{\mathbf{x}}^{(0)}(t_m)\big] > 0 \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha. \end{aligned} \tag{2.28}$$

PROOF. Equation (2.27) is identical to Eq. (2.19), thus the first condition in Eq. (2.19) is satisfied. From Definition 2.13, consider the boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta$ $(\beta \neq \alpha)$ first. Suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$ and $\mathbf{x}^{(\alpha)}(t)$ are $C^r_{[t_{m-\varepsilon}, t_m)}$- and $C^r_{(t_m, t_{m+\varepsilon}]}$-continuous $(r \geqslant 3)$ for time t and $\|\mathrm{d}^r\mathbf{x}^{(\alpha)}/\mathrm{d}t^r\| < \infty$ $(\alpha \in \{i, j\})$. For $a \in [t_{m-\varepsilon}, t_m)$ and $a \in (t_m, t_{m+\varepsilon}]$, the Taylor series expansion of $\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon})$ to $\mathbf{x}^{(\alpha)}(a)$ up to the third-order term gives

$$\begin{aligned} \mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon}) &\equiv \mathbf{x}^{(\alpha)}(t_{m\pm} - \varepsilon) \\ &= \mathbf{x}^{(\alpha)}(a) + \dot{\mathbf{x}}^{(\alpha)}(a)(t_{m\pm} \pm \varepsilon - a) + \ddot{\mathbf{x}}^{(\alpha)}(a)(t_{m\pm} \pm \varepsilon - a)^2 \\ &\quad + o\big((t_{m\pm} \pm \varepsilon - a)^2\big), \end{aligned}$$

As $a \to t_{m\pm}$, the limit of the foregoing equation leads to

$$\begin{aligned} \mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon}) &\equiv \mathbf{x}^{(\alpha)}(t_m \pm \varepsilon) \\ &= \mathbf{x}^{(\alpha)}(t_{m\pm}) \pm \dot{\mathbf{x}}^{(\alpha)}(t_{m\pm})\varepsilon + \ddot{\mathbf{x}}^{(\alpha)}(t_{m\pm})\varepsilon^2 + o\big(\varepsilon^2\big). \end{aligned}$$

The ignorance of the ε^3 and high order terms, the deformation of the above equation and left multiplication of $\mathbf{n}_{\partial\Omega_{ij}}$ gives

$$\begin{aligned} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})\big] &= \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+})\varepsilon + \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \ddot{\mathbf{x}}^{(\alpha)}(t_{m+})\varepsilon^2, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\big] &= \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-})\varepsilon - \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \ddot{\mathbf{x}}^{(\alpha)}(t_{m-})\varepsilon^2. \end{aligned}$$

With Eq. (2.27), we have

$$\begin{aligned} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})\big] &= \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \ddot{\mathbf{x}}^{(\alpha)}(t_{m+})\varepsilon^2, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\big] &= -\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \ddot{\mathbf{x}}^{(\alpha)}(t_{m-})\varepsilon^2. \end{aligned}$$

Because of $\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m\pm}) = 0$

$$\begin{aligned}
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathbf{x}^{(0)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+})\big] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \ddot{\mathbf{x}}^{(0)}(t_{m+})\varepsilon^2,\\
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathbf{x}^{(0)}(t_{m-}) - \mathbf{x}^{(0)}(t_{m-\varepsilon})\big] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \ddot{\mathbf{x}}^{(0)}(t_{m-})\varepsilon^2.
\end{aligned}$$

For the boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta$, using the first inequality of Eq. (2.28), the foregoing two equations lead to

$$\begin{aligned}
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\big]\\
&\quad = -\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\ddot{\mathbf{x}}^{(\alpha)}(t_{m-}) - \ddot{\mathbf{x}}^{(0)}(t_{m-})\big]\varepsilon^2 > 0,\\
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\big] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\ddot{\mathbf{x}}^{(\alpha)}(t_{m+}) - \ddot{\mathbf{x}}^{(0)}(t_{m+})\big]\varepsilon^2 < 0.
\end{aligned}$$

Similarly, using the second inequality of Eq. (2.28), the foregoing two equations for the boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha$ lead to

$$\begin{aligned}
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\big] = -\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\ddot{\mathbf{x}}^{(\alpha)}(t_{m-}) - \ddot{\mathbf{x}}^{(0)}(t_{m-})\big]\varepsilon^2 < 0,\\
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})\big] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\ddot{\mathbf{x}}^{(\alpha)}(t_{m+}) - \ddot{\mathbf{x}}^{(0)}(t_{m+})\big]\varepsilon^2 > 0.
\end{aligned}$$

Therefore under condition in Eq. (2.28), the flow $\mathbf{x}^{(t)}(t)$ in Ω_α is tangential to the boundary $\partial\Omega_{ij}$. □

THEOREM 2.11. *For a discontinuous dynamical system in Eq. (2.1), $\mathbf{x}(t_m) = \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$) and suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ ($\alpha \in \{i, j\}$). The vector field $\mathbf{F}^{(\alpha)}(t)$ is $C^r_{[t_{m-\varepsilon},t_m)}$- and $C^r_{(t_m,t_{m+\varepsilon}]}$-continuous ($r \geqslant 2$) for time t and $\|\mathrm{d}^{r+1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r+1}\| < \infty$. The flow $\mathbf{x}^{(\alpha)}(t)$ in Ω_α is tangential to the boundary $\partial\Omega_{ij}$ iff*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m\pm}) = 0 \quad \textit{and} \tag{2.29}$$

$$\begin{aligned}
&\textit{either} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathrm{D}\mathbf{F}^{(\alpha)}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}(t_{m\pm})\big] < 0 \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta\\
&\qquad (\beta \in \{i, j\}, \beta \neq \alpha),\\
&\textit{or} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathrm{D}\mathbf{F}^{(\alpha)}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}(t_{m\pm})\big] > 0 \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha
\end{aligned} \tag{2.30}$$

where the total differentiation ($p, q \in \{1, 2, \ldots, n\}$) and $\lambda \in \{0, \alpha\}$

$$\mathrm{D}\mathbf{F}^{(\lambda)}(t_{m\pm}) = \left[\frac{\partial F_p^{(\lambda)}(t_{m\pm})}{\partial x_q}\right]\mathbf{F}^{(\lambda)}(t_{m\pm}) + \frac{\partial \mathbf{F}^{(\lambda)}(t_{m\pm})}{\partial t}.$$

PROOF. Using Eqs. (2.1) and (2.29), the first condition in Eq. (2.19) is satisfied. The derivative of Eqs. (2.1) and (2.5) with respect to time gives

$$\ddot{\mathbf{x}} \equiv \left[\frac{\partial F_p^{(\lambda)}(\mathbf{x}, t, \boldsymbol{\mu}_\lambda)}{\partial x_q}\right]\dot{\mathbf{x}} + \frac{\partial}{\partial t}\mathbf{F}^{(\lambda)}(\mathbf{x}, t, \boldsymbol{\mu}_\lambda) = \mathrm{D}\mathbf{F}^{(\lambda)}(t_{m\pm}).$$

For $t = t_{m\pm}$ and $\mathbf{x} = \mathbf{x}_m$, the left multiplication of $\mathbf{n}_{\partial\Omega_{ij}}$ to the foregoing equation gives

$$\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \ddot{\mathbf{x}}(t_{m\pm}) = \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathrm{D}\mathbf{F}^{(\lambda)}(t_{m\pm})$$

where $\mathbf{F}^{(\alpha)}(\mathbf{x}_m, t_{m\pm}, \boldsymbol{\mu}_\alpha) \triangleq \mathbf{F}^{(\alpha)}(t_{m\pm})$. Using Eq. (2.30), the above equation leads to Eq. (2.28). From Theorem 2.10, the flow $\mathbf{x}^{(\alpha)}(t)$ in Ω_α is tangential to the boundary $\partial\Omega_{ij}$. □

DEFINITION 2.14. For a discontinuous dynamical system in Eq. (2.1), $\mathbf{x}(t_m) = \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$) and suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ ($\alpha \in \{i, j\}$). The flow $\mathbf{x}^{(\alpha)}(t)$ is $C^r_{[t_{m-\varepsilon}, t_m)}$- and $C^r_{(t_m, t_{m+\varepsilon}]}$-continuous ($r \geqslant 2l$) for time t. The flow $\mathbf{x}^{(\alpha)}(t)$ in Ω_α is the $(2l-1)$th-order tangential to the boundary $\partial\Omega_{ij}$ if the three conditions hold:

$$\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[\mathbf{x}^{(\alpha)}(t_{m\pm}) - \mathbf{x}^{(0)}(t_{m\pm})\right] = 0 \quad \text{for } k = 1, 2, \ldots, 2l-1, \tag{2.31}$$

$$\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \frac{\mathrm{d}^{2l}}{\mathrm{d}t^{2l}}\left[\mathbf{x}^{(\alpha)}(t_{m\pm}) - \mathbf{x}^{(0)}(t_{m\pm})\right] \neq 0, \quad \text{and} \tag{2.32}$$

either

$$\begin{aligned} &\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] > 0, \\ &\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] < 0 \\ &\quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta \end{aligned} \tag{2.33}$$

where $\beta \in \{i, j\}$ but $\alpha \neq \beta$, or

$$\begin{aligned} &\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] < 0, \\ &\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] > 0 \\ &\quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha. \end{aligned} \tag{2.34}$$

THEOREM 2.12. *For a discontinuous dynamical system in Eq. (2.1),* $\mathbf{x}(t_m) = \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$*) and suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ ($\alpha \in \{i, j\}$). *The flow* $\mathbf{x}^{(\alpha)}(t)$ *is* $C^r_{[t_{m-\varepsilon}, t_m)}$*- and* $C^r_{(t_m, t_{m+\varepsilon}]}$*-continuous* ($r \geqslant 2l+1$) *for time* t *and*

$\|\mathrm{d}^r \mathbf{x}^{(\alpha)}/\mathrm{d}t^r\| < \infty$. *The flow* $\mathbf{x}^{(\alpha)}(t)$ *in* Ω_α *is the* $(2l-1)$*th-order tangential to the boundary* $\partial\Omega_{ij}$ *iff*

$$\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \frac{\mathrm{d}^k}{\mathrm{d}t^k}\big[\mathbf{x}^{(\alpha)}(t_{m\pm}) - \mathbf{x}^{(0)}(t_{m\pm})\big] = 0 \quad \text{for } k = 1, 2, \ldots, 2l-1, \tag{2.35}$$

$$\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \frac{\mathrm{d}^{2l}}{\mathrm{d}t^{2l}}\big[\mathbf{x}^{(\alpha)}(t_{m\pm}) - \mathbf{x}^{(0)}(t_{m\pm})\big] \neq 0, \quad \text{and} \tag{2.36}$$

$$\left.\begin{aligned} &\text{either} \quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \frac{\mathrm{d}^{2l}}{\mathrm{d}t^{2l}}\big[\mathbf{x}^{(\alpha)}(t_{m\pm}) - \mathbf{x}^{(0)}(t_{m\pm})\big] < 0 \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta, \\ &\text{or} \quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \frac{\mathrm{d}^{2l}}{\mathrm{d}t^{2l}}\big[\mathbf{x}^{(\alpha)}(t_{m\pm}) - \mathbf{x}^{(0)}(t_{m\pm})\big] > 0 \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha \end{aligned}\right. \tag{2.37}$$

where $\beta \in \{i, j\}$ *but* $\alpha \neq \beta$.

PROOF. For Eqs. (2.35) and (2.36), the first two conditions in Definition 2.14 are satisfied. Consider the boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta$ ($\beta \neq \alpha$) first. Choose $a \in [t_{m-\varepsilon}, t_m)$ or $a \in (t_m, t_{m-\varepsilon}]$, and application of the Taylor series expansion of $\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon})$ to $\mathbf{x}^{(\alpha)}(a)$ and up to the $2l$-order term gives for $\lambda \in \{0, \alpha\}$

$$\begin{aligned} \mathbf{x}^{(\lambda)}(t_{m\pm\varepsilon}) &\equiv \mathbf{x}^{(\lambda)}(t_{m\pm} \pm \varepsilon) = \mathbf{x}^{(\lambda)}(a) + \sum_{k=1}^{2l-1} \frac{\mathrm{d}^k}{\mathrm{d}t^k}\mathbf{x}^{(\lambda)}(a)(t_{m\pm} \pm \varepsilon - a)^k \\ &\quad + \frac{\mathrm{d}^{2l}}{\mathrm{d}t^{2l}}\mathbf{x}^{(\lambda)}(a)(t_{m\pm} \pm \varepsilon - a)^{2l} + o\big((t_{m\pm} \pm \varepsilon - a)^{2l}\big). \end{aligned}$$

As $a \to t_{m\pm}$, we have

$$\begin{aligned} \mathbf{x}^{(\lambda)}(t_{m\pm\varepsilon}) &\equiv \mathbf{x}^{(\lambda)}(t_{m\pm} \pm \varepsilon) = \mathbf{x}^{(\lambda)}(t_{m\pm}) + \sum_{k=1}^{2l-1} \frac{\mathrm{d}^k}{\mathrm{d}t^k}\mathbf{x}^{(\lambda)}(t_{m\pm})(\pm\varepsilon)^k \\ &\quad + \frac{\mathrm{d}^{2l}}{\mathrm{d}t^{2l}}\mathbf{x}^{(\lambda)}(t_{m\pm})(\pm\varepsilon)^{2l} + o\big(\pm\varepsilon^{2l}\big). \end{aligned}$$

With Eqs. (2.35) and (2.36), the deformation of the above equation and left multiplication of $\mathbf{n}_{\partial\Omega_{ij}}$ produces

$$\begin{aligned} &\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \big[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\big] = \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \frac{\mathrm{d}^{2l}}{\mathrm{d}t^{2l}}\big[\mathbf{x}^{(\alpha)}(t_{m+}) - \mathbf{x}^{(0)}(t_{m+})\big]\varepsilon^{2l}, \\ &\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \big[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\big] \\ &\quad = -\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \frac{\mathrm{d}^{2l}}{\mathrm{d}t^{2l}}\big[\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(0)}(t_{m-})\big]\varepsilon^{2l}. \end{aligned}$$

Under Eq. (2.37), the condition in Eq. (2.33) is satisfied. Therefore, the flow $\mathbf{x}^{(\alpha)}(t)$ in Ω_α is the $(2l-1)$th-order tangential to the boundary $\partial\Omega_{\alpha\beta}$ with

$\mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta$. Similarly, under the condition in Eq. (2.37), the flow $\mathbf{x}^{(\alpha)}(t)$ in Ω_α is the $(2l-1)$th-order tangential to the boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha$. The theorem is proved. □

THEOREM 2.13. *For a discontinuous dynamical system in Eq. (2.1),* $\mathbf{x}(t_m) = \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$*) and suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$. *The vector field* $\mathbf{F}^{(\alpha)}(t)$ *is* $C^r_{[t_{m-\varepsilon}, t_m)}$*- and* $C^r_{(t_m, t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 2l)$ *for time t and* $\|\mathrm{d}^{r+1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r+1}\| < \infty$. *The flow* $\mathbf{x}^{(\alpha)}(t)$ *in* Ω_α *is the* $(2l-1)$*th-order tangential to the boundary* $\partial\Omega_{ij}$ *iff*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathrm{D}^{k-1}\big[\mathbf{F}^{(\alpha)}(t_{m\pm}) - \mathbf{F}^{(0)}(t_{m\pm})\big] = 0 \quad \text{for } k = 1, 2, \ldots, 2l-1, \tag{2.38}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathrm{D}^{2l-1}\big[\mathbf{F}^{(\alpha)}(t_{m\pm}) - \mathbf{F}^{(0)}(t_{m\pm})\big] \neq 0, \tag{2.39}$$

$$\left.\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathrm{D}^{2l-1}\big[\mathbf{F}^{(\alpha)}(t_{m\pm}) - \mathbf{F}^{(0)}(t_{m\pm})\big] < 0 \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta, \quad \text{or} \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathrm{D}^{2l-1}\big[\mathbf{F}^{(\alpha)}(t_{m\pm}) - \mathbf{F}^{(0)}(t_{m\pm})\big] > 0 \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha \end{aligned}\right. \tag{2.40}$$

where the total differentiation for $\lambda \in \{0, \alpha\}$

$$\mathrm{D}^{k-1}\mathbf{F}^{(\lambda)}(t_{m\pm}) = \mathrm{D}^{k-2}\left\{\left[\frac{\partial F_p^{(\lambda)}(t_{m\pm})}{\partial x_q}\right]\mathbf{F}^{(\lambda)}(t_{m\pm}) + \frac{\partial \mathbf{F}^{(\lambda)}(t_{m\pm})}{\partial t}\right\}, \tag{2.41}$$

$p, q \in \{1, 2, \ldots, n\}$, $k \in \{2, 3, \ldots, 2l\}$ *and* $\beta \in \{i, j\}$ *but* $\alpha \neq \beta$.

PROOF. The kth-order derivative of Eq. (2.1) with respect to time gives

$$\begin{aligned} \frac{\mathrm{d}^k\mathbf{x}^{(\lambda)}(t_m)}{\mathrm{d}t^k} &= \frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}\dot{\mathbf{x}}^{(\lambda)}(t_m) = \frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}\mathbf{F}^{(\lambda)}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_\alpha) \\ &\equiv \mathrm{D}^{k-1}\mathbf{F}^{(\lambda)}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_\alpha) \\ &= \mathrm{D}^{k-2}\left\{\left[\frac{\partial F_p^{(\lambda)}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_\alpha)}{\partial x_q}\right]\dot{\mathbf{x}} + \frac{\partial}{\partial t}\mathbf{F}^{(\lambda)}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_\alpha)\right\}. \end{aligned}$$

Using the foregoing equation to the conditions in Eqs. (2.38)–(2.41), the flow $\mathbf{x}^{(\alpha)}(t)$ in Ω_α is the $(2l-1)$th-order tangential to the boundary $\partial\Omega_{ij}$ from Theorem 2.12. Therefore, the theorem is proved. □

DEFINITION 2.15. A flow $\mathbf{x}^{(\alpha)}(t)$ tangential to $\partial\Omega_{ij}$ $(\alpha \in \{i, j\})$ in Ω_α is termed the local grazing flow if $\mathbf{x}^{(\alpha)}(t)$ starting from $\partial\Omega_{ij}$ in Ω_α is not intersected with another boundary before grazing. Suppose $\mathbf{x}^{(\alpha)}(t)$ has $\mathbf{x}^{(\alpha)}_{m-1}$ and $\mathbf{x}^{(\alpha)}_{m+1}$ on the $\mathbf{n}_{\partial\Omega_{ij}}$-line relative to $\mathbf{x}_m \in \partial\Omega_{ij}$, then the three grazing flows exist. The tangential flow $\mathbf{x}^{(\alpha)}(t)$ is termed:

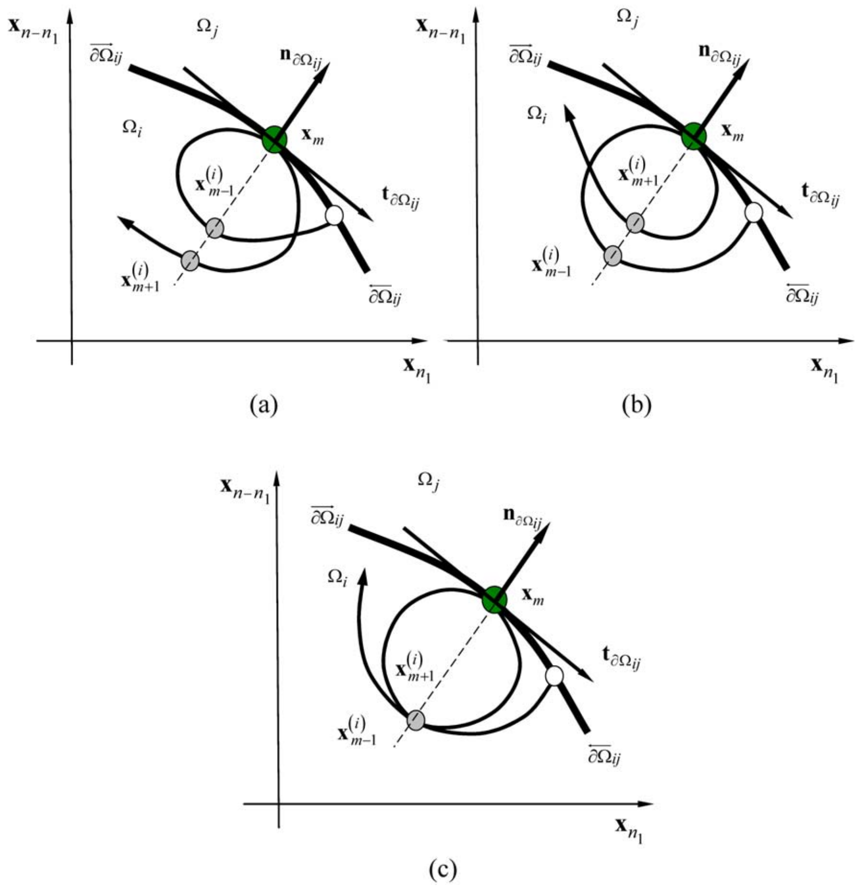


Figure 2.8. A classification of local grazing flows in Ω_i to $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$: (a) the first kind of grazing flow, (b) the second kind of grazing flow, and (c) the third kind of grazing flow. The gray-filled symbols represent two points ($\mathbf{x}^{(i)}_{m-1}$ and $\mathbf{x}^{(i)}_{m+1}$) on the normal line relative to tangential point $\mathbf{x}_m$ on the boundary $\partial\Omega_{ij}$ depicted by a large circular symbol.

(a) *the grazing flow of the first kind* if

$$\left\|\mathbf{x}^{(\alpha)}_{m-1} - \mathbf{x}_m\right\| < \left\|\mathbf{x}^{(\alpha)}_{m+1} - \mathbf{x}_m\right\|; \tag{2.42}$$

(b) *the grazing flow of the second kind* if

$$\left\|\mathbf{x}^{(\alpha)}_{m-1} - \mathbf{x}_m\right\| > \left\|\mathbf{x}^{(\alpha)}_{m+1} - \mathbf{x}_m\right\|; \tag{2.43}$$

(c) *the grazing flow of the third kind* if

$$\left\|\mathbf{x}_{m-1}^{(\alpha)}-\mathbf{x}_m\right\|=\left\|\mathbf{x}_{m+1}^{(\alpha)}-\mathbf{x}_m\right\|. \tag{2.44}$$

From the above definition, the local grazing flows in the domain Ω_i to the boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}}\to\Omega_j$ are sketched in Fig. 2.8 for interpretation of the local grazing flows. The first, second and third kinds of grazing flow are arranged in Figs. 2.8(a), (b) and (c), respectively. The gray-filled symbols represent two points ($\mathbf{x}_{m-1}^{(i)}$ and $\mathbf{x}_{m+1}^{(i)}$) on the normal line relative to tangential point $\mathbf{x}_m$ on the boundary $\partial\Omega_{ij}$ depicted by a large circular symbol.

In this section, the sufficient and necessary conditions presented for grazing, passable and nonpassable flows are based on the function $\varphi_{ij}(\mathbf{x}(t))=0$. If the normal direction of the boundary $\partial\Omega_{ij}$ is invariant with time t, then we have $\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\cdot\frac{\mathrm{d}^k}{\mathrm{d}t^k}\mathbf{x}^{(0)}(t_{m\pm})=0$ for $k=1,2,\ldots$. If the separation boundary is controlled by $\varphi_{ij}(\mathbf{x},t)=0$, all the conditions $\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\cdot\dot{\mathbf{x}}^{(\alpha)}(t_{m\pm})=0$ (or $\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\cdot\mathbf{F}^{(\alpha)}(t_{m\pm})=0$) for the separation boundary $\varphi_{ij}(\mathbf{x}(t))=0$ are replaced by $\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\cdot\dot{\mathbf{x}}^{(\alpha)}(t_{m\pm})+\frac{\partial\varphi_{ij}}{\partial t}=0$ and $\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\cdot\mathbf{F}^{(\alpha)}(t_{m\pm})+\frac{\partial\varphi_{ij}}{\partial t}=0$, respectively. The rest conditions can be kept as in the present form. In an alternative way, a new function $G(\mathbf{x},t)=\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\cdot\dot{\mathbf{x}}^{(\alpha)}(t_{m\pm})+\frac{\partial\varphi_{ij}}{\partial t}$ (or $G(\mathbf{x},t)=\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\cdot\mathbf{F}(t_{m\pm})+\frac{\partial\varphi_{ij}}{\partial t}$) is introduced. We have $\mathrm{D}^kG(\mathbf{x},t)=\mathrm{D}^{k-1}(\mathrm{D}G(\mathbf{x},t))$ and $\mathrm{D}G(\mathbf{x},t)=\nabla G(\mathbf{x},t)\cdot\dot{\mathbf{x}}^{(\alpha)}(t_{m\pm})+\frac{\partial G(\mathbf{x},t)}{\partial t}$ (or $\mathrm{D}G(\mathbf{x},t)=\nabla G(\mathbf{x},t)\cdot\mathbf{F}^{(\alpha)}(t_{m\pm})+\frac{\partial G(\mathbf{x},t)}{\partial t}$). The necessary and sufficient conditions for grazing require $G(\mathbf{x},t)=0$ and $\mathrm{D}G(\mathbf{x},t)\neq 0$. The other conditions for higher-order grazing and transversality will use $\mathrm{D}^{k-1}G(\mathbf{x},t)$ to replace $\frac{\mathrm{d}^k}{\mathrm{d}t^k}(\mathbf{x}^{(\alpha)}(t)-\mathbf{x}^{(0)}(t))$ or $\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}(\mathbf{F}^{(\alpha)}(t)-\mathbf{F}^{(0)}(t))$. We will use the separation boundary condition $\varphi_{ij}(\mathbf{x}(t))=0$ to develop the theory.

2.5. A piecewise linear system

To demonstrate the local singularity and grazing in nonsmooth dynamical systems, two common examples in mechanical engineering are considered herein. Consider a periodically excited, piecewise linear system as the first application in this section with

$$\ddot{x}+2d\dot{x}+k(x)=a\cos\Omega t, \tag{2.45}$$

where $\dot{x}=\mathrm{d}x/\mathrm{d}t$. The parameters ($\Omega$ and a) are excitation frequency and amplitude, respectively. The restoring force is

$$k(x)=\begin{cases} cx-e, & \text{for } x\in[E,\infty),\\ 0, & \text{for } x\in[-E,E],\\ cx+e, & \text{for } x\in(-\infty,-E] \end{cases} \tag{2.46}$$

with $E = e/c$. To describe the motion of the foregoing system, there are three linear regions of the restoring force (region I: $x \geqslant E$, region II: $-E \leqslant x \leqslant E$ and region III: $x \leqslant -E$). The solution for each region can be easily obtained in Appendix A. From Eqs. (2.45) and (2.46), the dynamical systems in regions I and III do not have any singularity. Once all the parameters are finite, the solutions of these motions in regions I and III are finite. In region II, the solution of the motion continuous to time t is unbounded, but the displacement boundary of the domain on which the dynamical system is defined is finite. Since the velocity is the derivative of displacement with respect to time, the flows of the system in the displacement-bounded, region II is bounded. The phase space in Eq. (2.45) is divided into three subdomains, and the three subdomains are defined by

$$\begin{aligned} \Omega_1 &= \left\{(x, y) \mid x \in [E, \infty), y \in (-\infty, \infty)\right\}, \\ \Omega_2 &= \left\{(x, y) \mid x \in [-E, E], y \in (-\infty, \infty)\right\}, \\ \Omega_3 &= \left\{(x, y) \mid x \in (-\infty, -E], y \in (-\infty, \infty)\right\}. \end{aligned} \tag{2.47}$$

The entire phase space is given by

$$\Omega = \bigcup_{\alpha=1}^{3} \Omega_\alpha. \tag{2.48}$$

The corresponding separation boundaries are

$$\begin{aligned} \partial\Omega_{12} &= \Omega_1 \cap \Omega_2 = \left\{(x, y) \mid \varphi_{12}(x, y) \equiv x - E = 0\right\}, \\ \partial\Omega_{23} &= \Omega_2 \cap \Omega_3 = \left\{(x, y) \mid \varphi_{23}(x, y) \equiv x + E = 0\right\}. \end{aligned} \tag{2.49}$$

Such domains and the boundary are sketched in Fig. 2.9. From the above definitions, Eqs. (2.45) and (2.46) give

$$\mathbf{x} = \mathbf{F}^{(\alpha)}(\mathbf{x}, t, \boldsymbol{\mu}_\alpha, \boldsymbol{\pi}) \quad \text{for } \alpha \in \{1, 2, 3\} \tag{2.50}$$

where

$$\begin{aligned} &\mathbf{F}^{(1)}(\mathbf{x}, t, \boldsymbol{\mu}_1, \boldsymbol{\pi}) = (y, -2dy - cx + a\cos\omega t)^{\mathrm{T}}, && \text{in } \mathbf{x} \in \Omega_1 \\ &\mathbf{F}^{(2)}(\mathbf{x}, t, \boldsymbol{\mu}_2, \boldsymbol{\pi}) = (y, -2dy + a\cos\omega t)^{\mathrm{T}}, && \text{in } \mathbf{x} \in \Omega_2, \\ &\mathbf{F}^{(3)}(\mathbf{x}, t, \boldsymbol{\mu}_3, \boldsymbol{\pi}) = (y, -2dy - cx + a\cos\omega t)^{\mathrm{T}}, && \text{in } \mathbf{x} \in \Omega_3. \end{aligned} \tag{2.51}$$

Note that $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_3 = (c, d)^{\mathrm{T}}$, $\boldsymbol{\mu}_2 = (0, d)^{\mathrm{T}}$ and $\boldsymbol{\pi} = (\Omega, a)^{\mathrm{T}}$. To understand the local dynamics of Eq. (2.45), it is very important to investigate the sliding dynamics along the separation boundary. In Eq. (2.49), using $\mathbf{n}_{\partial\Omega_{ij}} = \nabla\varphi_{ij}$ gives

$$\mathbf{n}_{\partial\Omega_{12}} = \mathbf{n}_{\partial\Omega_{23}} = (1, 0)^{\mathrm{T}}. \tag{2.52}$$

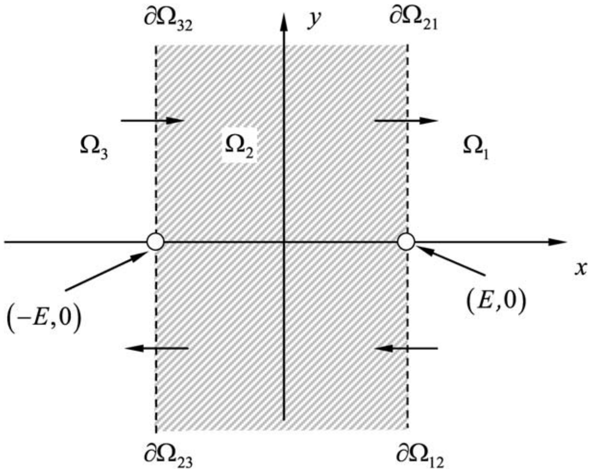


Figure 2.9. Subdomains, boundaries and equilibriums $(\pm E, 0)$.

Therefore, Eqs. (2.51) and (2.52) give

$$\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot \mathbf{F}^{(1)}(\mathbf{x}, t, \boldsymbol{\mu}_1, \boldsymbol{\pi}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot \mathbf{F}^{(2)}(\mathbf{x}, t, \boldsymbol{\mu}_2, \boldsymbol{\pi}) = y, \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{23}} \cdot \mathbf{F}^{(2)}(\mathbf{x}, t, \boldsymbol{\mu}_2, \boldsymbol{\pi}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{23}} \cdot \mathbf{F}^{(3)}(\mathbf{x}, t, \boldsymbol{\mu}_3, \boldsymbol{\pi}) = y. \end{aligned} \tag{2.53}$$

From Eqs. (2.52) and (2.53) with Theorem 2.11, the equilibrium points $(\pm E, 0)$ are common tangential points for flows in both subdomains of the two separatrices. Furthermore, the grazing bifurcation conditions on the separatrices for the flows of this discontinuous system in the three subdomains are

$$y = 0 \quad \text{and} \quad \begin{cases} \dot{y} > 0 & \text{for } x = E \text{ in } \Omega_1 \text{ and } x = -E \text{ in } \Omega_2, \\ \dot{y} < 0 & \text{for } x = E \text{ in } \Omega_2 \text{ and } x = -E \text{ in } \Omega_3. \end{cases} \tag{2.54}$$

Therefore, in the neighborhoods of the two equilibrium points, the local topological structures of the hyperbolic flow for system in Eq. (2.45) are sketched in Fig. 2.10. The detailed discussion can be refereed to Luo (2006). For description of motion in Eq. (2.45), two switching sections (or sets) are:

$$\begin{aligned} &\Sigma^+ = \big\{(t_i, x_i, y_i) \mid x_i = E, \dot{x}_i = y_i\big\}, \\ &\Sigma^- = \big\{(t_i, x_i, y_i) \mid x_i = -E, \dot{x}_i = y_i\big\}. \end{aligned} \tag{2.55}$$

The two sets are decomposed as

$$\Sigma^+ = \Sigma^+_+ \cup \Sigma^+_- \cup \{t_i, E, 0\} \quad \text{and} \quad \Sigma^- = \Sigma^-_+ \cup \Sigma^-_- \cup \{t_i, -E, 0\} \tag{2.56}$$

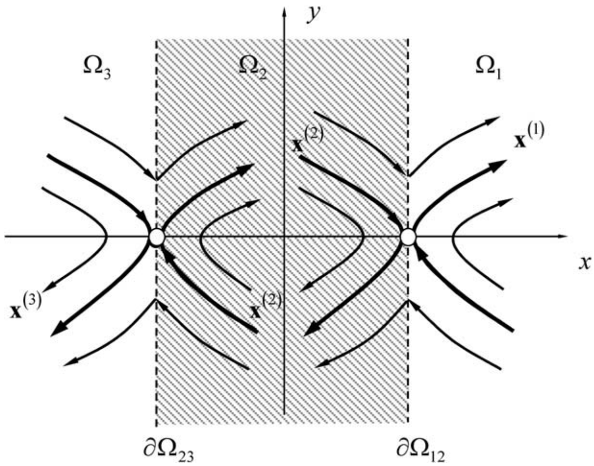


Figure 2.10. Phase portraits near equilibriums $(\pm E, 0)$ on the boundaries.

where four subsets are defined as

$$\begin{aligned} \Sigma_+^+ &= \left\{(t_i, x_i, y_i) \mid x_i = E, \dot{x}_i = y_i > 0\right\}, \\ \Sigma_-^+ &= \left\{(t_i, x_i, y_i) \mid x_i = E, \dot{x}_i = y_i < 0\right\}, \end{aligned} \tag{2.57}$$

$$\begin{aligned} \Sigma_+^- &= \left\{(t_i, x_i, y_i) \mid x_i = -E, \dot{x}_i = y_i > 0\right\}, \\ \Sigma_-^- &= \left\{(t_i, x_i, y_i) \mid x_i = -E, \dot{x}_i = y_i < 0\right\}. \end{aligned} \tag{2.58}$$

From four subsets, six basic mappings are:

$$\begin{aligned} &P_1 : \Sigma_+^+ \to \Sigma_-^+, \quad P_2 : \Sigma_-^+ \to \Sigma_-^-, \quad P_3 : \Sigma_-^- \to \Sigma_+^-, \\ &P_4 : \Sigma_+^- \to \Sigma_+^+, \quad P_5 : \Sigma_-^+ \to \Sigma_+^+, \quad P_6 : \Sigma_+^- \to \Sigma_-^-. \end{aligned} \tag{2.59}$$

The switching planes and basic mappings are sketched in Fig. 2.11. Consider the initial and final states of $(t, x, \dot{x})$ to be (t_i, x_i, y_i) and $(t_{i+1}, x_{i+1}, y_{i+1})$ in the subdomain Ω_α ($\alpha = 1, 2, 3$), respectively. The local mappings are $\{P_1, P_3, P_5, P_6\}$ and the global mappings are $\{P_2, P_4\}$. The displacement and velocity equations in Appendix A with initial conditions give the governing equations for mapping P_k ($k = 1, 2, \ldots, 6$), i.e.

$$P_k: \quad \begin{cases} f_1^{(k)}(x_i, y_i, t_i, x_{i+1}, y_{i+1}, t_{i+1}) = 0, \\ f_2^{(k)}(x_i, y_i, t_i, x_{i+1}, y_{i+1}, t_{i+1}) = 0. \end{cases} \tag{2.60}$$

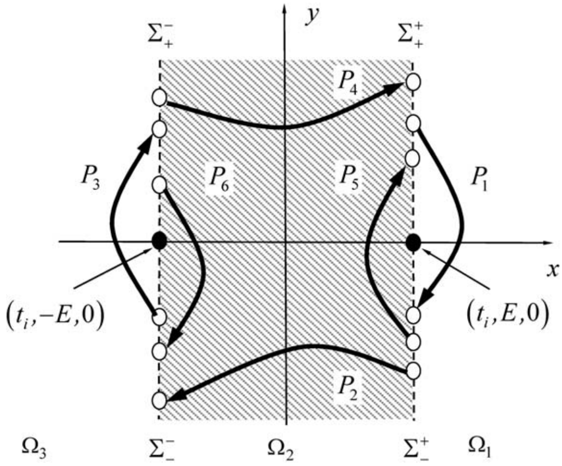


Figure 2.11. Switching sections and basic mappings in phase plane.

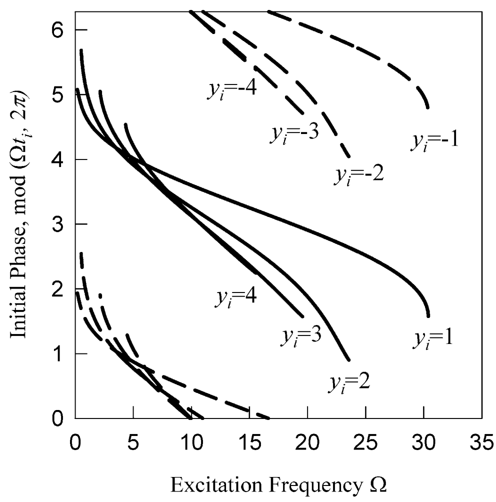


Figure 2.12. The grazing bifurcation conditions for mappings P_1 ($y_i \geqslant 0$, solid lines) and P_3 ($y_i \leqslant 0$, dashed lines). ($n = 1$, $a = 20$, $c = 100$, $E = 1$, $d = 0.5$.)

The necessary and sufficient conditions for the grazing of all the six generic mappings are:

$$
\begin{aligned}
&y_{i+1} = 0,\\
&\dot{y}_{i+1} = a\cos\Omega t_{i+1} > 0 \quad \text{for } P_j \ (j = 1, 2, 6),\\
&\dot{y}_{i+1} = a\cos\Omega t_{i+1} < 0 \quad \text{for } P_j \ (j = 3, 4, 5).
\end{aligned}
\tag{2.61}
$$

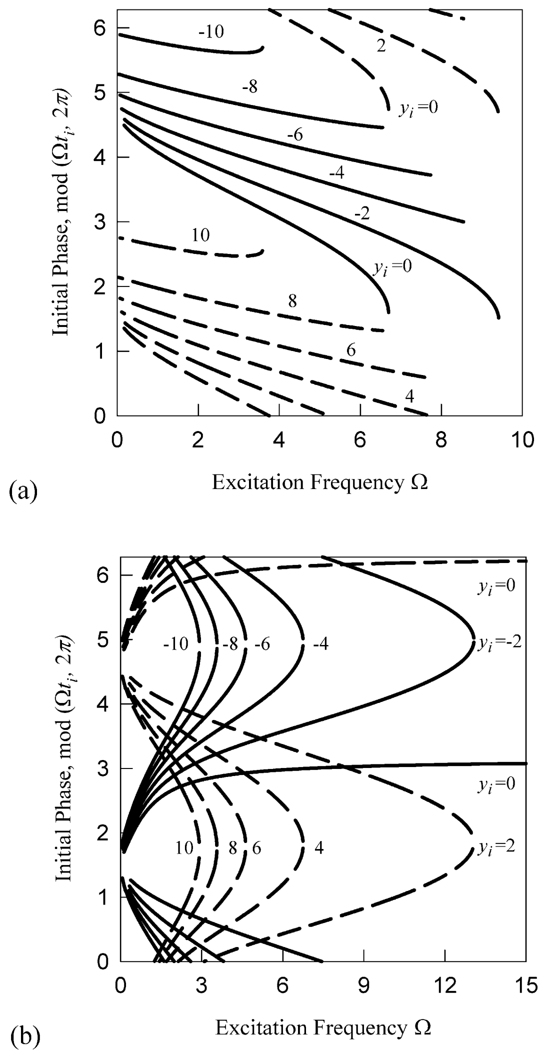


Figure 2.13. The grazing bifurcation conditions for: (a) mappings P_2 ($y_i \leqslant 0$, solid lines) and P_4 ($y_i \geqslant 0$, dashed lines), and (b) mappings P_5 ($y_i \leqslant 0$, solid lines) and P_6 ($y_i \geqslant 0$, dashed lines). ($n = 1, a = 20, c = 100, E = 1, d = 0.5$.)

From the foregoing equation, once one of the initial time or velocity is selected, the grazing bifurcation can be determined through the Newton–Raphson method. In computation, the condition $t_{i+1} > t_i$ should be inserted for fast obtaining solutions. Consider the parameters $a = 20, c = 100, E = 1, d = 0.5$ for illustration of the grazing bifurcation. For given initial velocity y_i, the grazing bifurcations

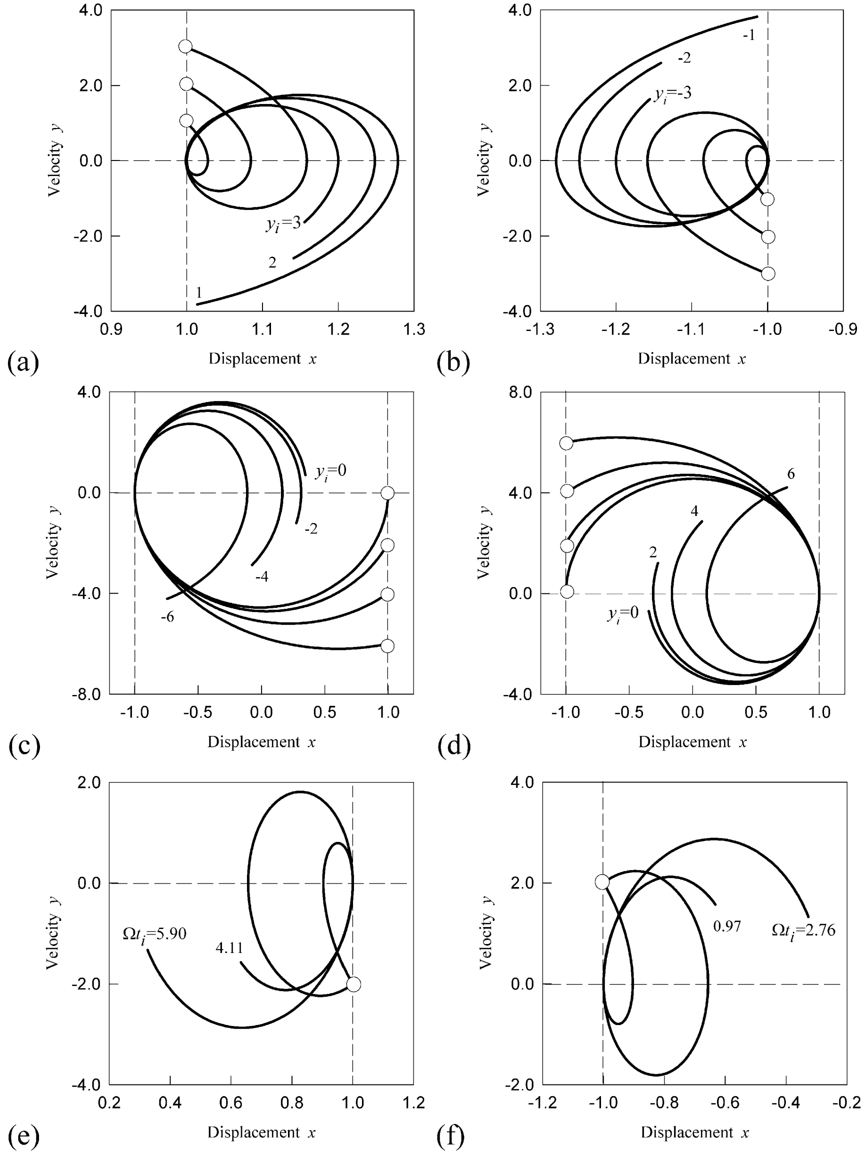


Figure 2.14. The grazing flows relative to: (a) mapping P_1 $(x_i = 1)$, (b) mapping P_3 $(x_i = -1)$, (c) mapping P_2 $(x_i = -1)$, (d) mapping P_4 $(x_i = 1)$, (e) mapping P_5 $(x_i = 1)$, and (f) mapping P_6 $(x_i = -1)$. $(n = 1, a = 20, c = 100, E = 1, d = 0.5.)$

are computed. In Fig. 2.12, the grazing bifurcation conditions of mapping P_1 ($y_i \geqslant 0$) and P_3 ($y_i \leqslant 0$) are illustrated for $y_i = \pm\{1, 2, 3, 4\}$ and $n = 1$. The grazing bifurcations for both mappings P_2 ($y_i \leqslant 0$) and P_4 ($y_i \geqslant 0$), and mappings P_2 ($y_i \leqslant 0$) and P_4 ($y_i \geqslant 0$) are presented in Figs. 2.13(a) and (b). The mappings $\{P_2, P_5\}$ and $\{P_3, P_6\}$ are represented by the solid and dashed curves, respectively. The initial conditions varying with excitation frequency are presented. Similarly, the initial condition varying with excitation amplitude can be determined. From these initial conditions, based on the six mappings, the final states will be tangential to the corresponding boundaries. From the predictions of grazing bifurcation, the grazing flows are demonstrated for a better understanding of the grazing bifurcation. The same parameters are used as before. The grazing flows for mapping P_1 ($x_i = 1$, $(y_i, \Omega t_i) \approx (1, 3.592), (2, 3.256)$ and $(3, 3.125)$) in Ω_1 and mapping P_3 ($x_i = -1$, $(y_i, \Omega t_i) \approx (-1, 0.449), (-2, 0.114)$ and $(-3, 6.267)$) in Ω_3 are illustrated in Figs. 2.14(a) and (b) for $\Omega = 10$. The grazing flows in Ω_2 are illustrated in Figs. 2.14(c) and (d) for mapping P_2 ($x_i = 1$, $(y_i, \Omega t_i) \approx (0, 2.71), (-2, 3.20), (-4, 3.63)$ and $(-6, 4.07)$) and mapping P_4 ($x_i = -1$, $(y_i, \Omega t_i) \approx (0, 5.85), (2, 0.06), (-4, 0.47)$ and $(-6, 0.93)$). The grazing flow for $y_i = 0$ seems the half generic separatrix of the pendulum. Finally, the local grazing flows in domain Ω_2 are illustrated in Fig. 2.14(e) and (f) for mappings P_5 ($x_i = 1$, $y_i = -2$, $\Omega t_i \approx \{5.90, 4.11\}$) and mapping P_6 ($x_i = -1$, $y_i = 2$, $\Omega t_i \approx \{2.76, 0.97\}$). The two local, grazing flows are of the first kind in domain Ω_2. It is observed that the grazing flows at the tangential points satisfy the conditions in Theorem 2.11. The detailed results can be referred to Luo (2005b) The fragmentation of strange attractors of chaotic motions in this system can be seen in Luo (2005c). The grazing in piecewise linear systems with impacting can be found in Luo and Chen (2006).

2.6. A friction-induced oscillator

For the second example, consider a periodically forced oscillator consisting of a mass (m), a spring of stiffness (k) and a damper of viscous damping coefficient (r), as shown in Fig. 2.15(a). The grazing flow for this problem was discussed in Luo and Gegg (2006c). This oscillator also rests on the horizontal belt surface traveling with a constant speed (V). The absolute coordinate system (x, t) is for the mass. Consider a periodical force $Q_0 \cos \Omega t$ exerting on the mass, where Q_0 and Ω are the excitation strength and frequency, respectively. Since the mass contacts the moving belt with friction, the mass can move along or rest on the belt surface. Once the nonstick motion exists, a kinetic friction force shown in Fig. 2.15(b) is described as

$$\overline{F}_f(\dot{x}) \begin{cases} = \mu_k F_N, & \dot{x} \in [V, \infty), \\ \in [-\mu_k F_N, \mu_k F_N], & \dot{x} = V, \\ = -\mu_k F_N, & \dot{x} \in (-\infty, V] \end{cases} \tag{2.62}$$

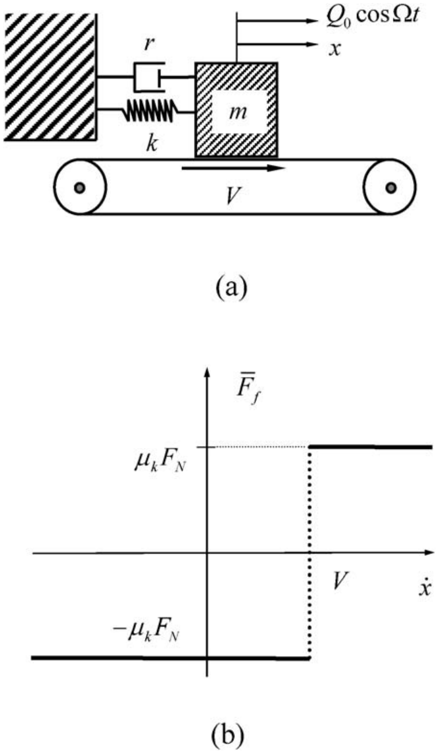


Figure 2.15. (a) Schematic of mechanical model, and (b) friction force.

where $\dot{x} \triangleq \mathrm{d}x/\mathrm{d}t$, μ_k and F_N are friction coefficient and a normal force to the contact surface, respectively. For the model in Fig. 2.15, the friction force is $F_N = mg$ where g is the gravitational acceleration.

For the mass moving with the same speed of the belt surface, the nonfriction force per unit mass acting on the mass in the x-direction during this motion is defined as

$$F_s = A_0 \cos\Omega t - 2dV - cx, \quad \text{for } \dot{x} = V \tag{2.63}$$

where $A_0 = Q_0/m, d = r/2m$ and $c = k/m$. This force cannot overcome the friction force during the stick motion (i.e. $|F_s| \leqslant F_f$ and $F_f = \mu_k F_N/m$). Therefore, the mass does not have any relative motion to the belt. No acceleration exists, i.e.,

$$\ddot{x} = 0, \quad \text{for } \dot{x} = V. \tag{2.64}$$

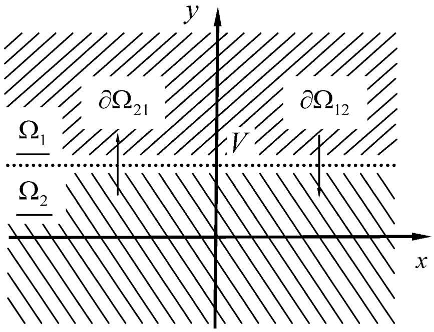


Figure 2.16. Domain partitions in phase plane.

If $|F_s| > F_f$, the nonfriction force will overcome the static friction force on the mass and the nonstick motion will appear. For the nonstick motion, the total force acting on the mass is

$$F = A_0 \cos \Omega t - F_f \operatorname{sgn}(\dot{x} - V) - 2d\dot{x} - cx, \quad \text{for } \dot{x} \neq V; \tag{2.65}$$

sgn(·) is the sign function. Therefore, the equation of the nonstick motion for this oscillator with friction is

$$\ddot{x} + 2d\dot{x} + cx = A_0 \cos \Omega t - F_f \operatorname{sgn}(\dot{x} - V), \quad \text{for } \dot{x} \neq V. \tag{2.66}$$

Since the friction force is dependent on the direction of the relative velocity, the phase plane is partitioned into two regions in which the motion is described through the continuous dynamical systems, as shown in Fig. 2.16. The two regions are expressed by Ω_α ($\alpha \in \{1, 2\}$). In phase plane, the following vectors are introduced as

$$\mathbf{x} \triangleq (x, \dot{x})^{\mathrm{T}} \equiv (x, y)^{\mathrm{T}} \quad \text{and} \quad \mathbf{F} \triangleq (y, F)^{\mathrm{T}}. \tag{2.67}$$

The mathematical description of the regions and boundary

$$\begin{aligned} &\Omega_1 = \{(x, y) \mid y \in (V, \infty)\}, \quad \Omega_2 = \{(x, y) \mid y \in (-\infty, V)\}, \\ &\partial\Omega_{12} = \{(x, y) \mid \varphi_{12}(x, y) \equiv y - V = 0\}, \\ &\partial\Omega_{21} = \{(x, y) \mid \varphi_{21}(x, y) \equiv y - V = 0\}. \end{aligned} \tag{2.68}$$

The subscript $(\cdot)_{\alpha\beta}$ denotes the boundary from Ω_α to Ω_β ($\alpha, \beta \in \{1, 2\}$ and $\alpha \neq \beta$). The equations of motion in Eqs. (2.64) and (2.66) can be described as

$$\dot{\mathbf{x}} = \mathbf{F}^{(\alpha)}(\mathbf{x}, t) \quad \text{in } \Omega_\alpha \ (\alpha \in \{1, 2\}), \tag{2.69}$$

where

$$\mathbf{F}^{(\alpha)}(\mathbf{x}, t) = \big(y, F_\alpha(\mathbf{x}, \Omega t)\big)^{\mathrm{T}}, \tag{2.70}$$

$$F_\alpha(\mathbf{x}, \Omega t) = A_0 \cos \Omega t - b_\alpha - 2d_\alpha y - c_\alpha x. \tag{2.71}$$

Note that $b_1 = \mu g$, $b_2 = -\mu g$, $d_\alpha = d$ and $c_\alpha = c$ for the model in Fig. 2.15. From Theorem 2.11, with $\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{DF}^{(0)} = 0$ the grazing motion is guaranteed by

$$\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(t_{m\pm}) = 0, \quad \alpha \in \{1, 2\}, \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{DF}^{(1)}(t_{m\pm}) > 0, \qquad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{DF}^{(2)}(t_{m\pm}) < 0, \end{aligned} \tag{2.72}$$

where

$$\mathbf{DF}^{(\alpha)}(t) = \left(2F_\alpha(\mathbf{x}, t), \nabla F_\alpha(\mathbf{x}, t) \cdot \mathbf{F}^{(\alpha)}(t) + \frac{\partial F_\alpha(\mathbf{x}, t)}{\partial t}\right)^{\mathrm{T}} \tag{2.73}$$

with $\nabla = \partial/\partial x \mathbf{i} + \partial/\partial y \mathbf{j}$ being the Hamilton operator. The time t_m represents the time for the motion on the velocity boundary. $t_{m\pm} = t_m \pm 0$ reflects the responses on the regions rather than boundary. Using the third and fourth equations of Eq. (2.68), the normal vector of the boundary are

$$\mathbf{n}_{\partial\Omega_{12}} = \mathbf{n}_{\partial\Omega_{21}} = (0, 1)^{\mathrm{T}}. \tag{2.74}$$

Therefore, we have

$$\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(t) = F_\alpha(\mathbf{x}, \Omega t), \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{DF}^{(\alpha)}(t) = \nabla F_\alpha(\mathbf{x}, \Omega t) \cdot \mathbf{F}^{(\alpha)}(t) + \frac{\partial F_\alpha(\mathbf{x}, \Omega t)}{\partial t}. \end{aligned} \tag{2.75}$$

From Eqs. (2.74) and (2.75), the necessary and sufficient conditions for grazing motions are from Theorem 2.10:

$$F_\alpha(\mathbf{x}_m, \Omega t_{m\pm}) = 0, \qquad F_\alpha(\mathbf{x}_m, \Omega t_{m-\varepsilon}) F_\alpha(\mathbf{x}_m, \Omega t_{m+\varepsilon}) < 0 \tag{2.76}$$

or precisely,

$$\begin{aligned} &F_\alpha(\mathbf{x}_m, \Omega t_{m\pm}) = 0, \\ &F_1(\mathbf{x}_m, \Omega t_{m-\varepsilon}) < 0, \qquad F_1(\mathbf{x}_m, \Omega t_{m+\varepsilon}) > 0, \\ &F_2(\mathbf{x}_m, \Omega t_{m-\varepsilon}) > 0, \qquad F_2(\mathbf{x}_m, \Omega t_{m+\varepsilon}) < 0. \end{aligned} \tag{2.77}$$

However, from Theorem 2.11, the necessary and sufficient conditions for grazing is given by

$$\begin{aligned} &F_\alpha(\mathbf{x}_m, \Omega t_{m\pm}) = 0, \\ &\nabla F_\alpha(\mathbf{x}_m, \Omega t_{m\pm}) \cdot \mathbf{F}^{(\alpha)}(t_{m\pm}) + \frac{\partial F_\alpha(\mathbf{x}_m, \Omega t_{m\pm})}{\partial t} \begin{cases} > 0, & \text{for } \alpha = 1, \\ < 0, & \text{for } \alpha = 2. \end{cases} \end{aligned} \tag{2.78}$$

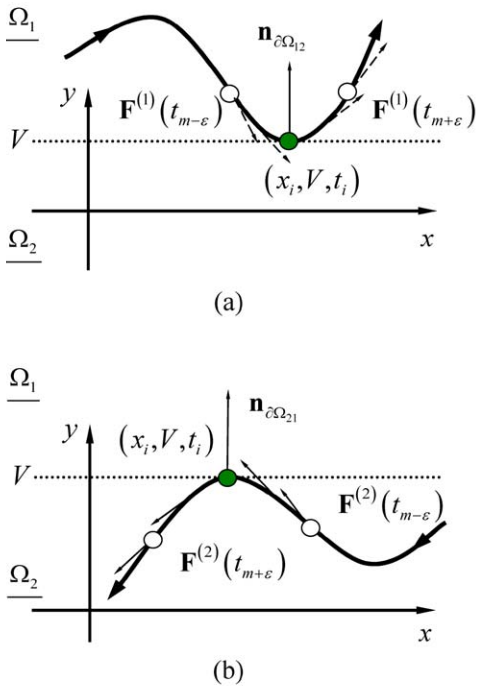


Figure 2.17. Vector fields for grazing motion in Ω_α ($\alpha = 1, 2$).

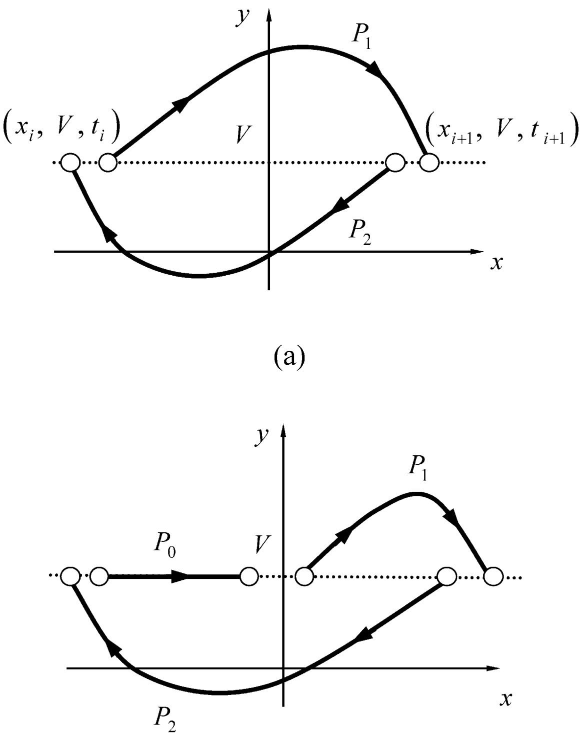
A sketch of grazing motions in domain Ω_α ($\alpha = 1, 2$) is illustrated in Figs. 2.17(a) and (b). The grazing conditions are also presented, and the vector fields in Ω_1 and Ω_2 are expressed by the dashed and solid arrow-lines, respectively. The condition in Eq. (2.76) for the grazing motion in Ω_α is presented through the vector fields of $\mathbf{F}^{(\alpha)}(t)$. In addition to $F_\alpha(\mathbf{x}_m, t_{m\pm}) = 0$, the sufficient condition requires $F_1(\mathbf{x}_m, \Omega t_{m-\varepsilon}) < 0$ and $F_1(\mathbf{x}_m, \Omega t_{m+\varepsilon}) > 0$ in domain Ω_1; and $F_2(\mathbf{x}_m, \Omega t_{m-\varepsilon}) > 0$ and $F_2(\mathbf{x}_m, \Omega t_{m+\varepsilon}) < 0$ in domain Ω_2. The detailed discussion can be refereed to Luo and Gegg (2006c).

After grazing motion, the sliding motion will appear. Direct integration of Eq. (2.64) with initial condition (t_i, x_i, V) gives the sliding motion, i.e.,

$$x = V(t - t_i) + x_i. \tag{2.79}$$

Substitution of Eq. (2.79) into (2.71) gives the forces for the very small δ-neighborhood of the stick motion ($\delta \to 0$) in the two domains Ω_α ($\alpha \in \{1, 2\}$), i.e.,

$$F_\alpha(\mathbf{x}_m, \Omega t_{m-}) = -2d_\alpha V - c_\alpha\big[V(t_m - t_i) + x_i\big] + A_0 \cos \Omega t_m - b_\alpha. \tag{2.80}$$



(a)

(b)

Figure 2.18. Regular mappings (P_1 and P_2) and stick mapping P_0.

For the nonstick motion, select the initial condition on the velocity boundary (i.e., $\dot{x}_i = V$), and then the coefficients of the solution in Appendix A, $C_k^{(\alpha)}(x_i, \dot{x}_i, t_i) \triangleq C_k^{(\alpha)}(x_i, t_i)$ for $k = 1, 2$. The basic solutions in Appendix A will be used for construction of mappings.

In phase plane, the trajectories in Ω_α starting and ending at the velocity boundary (i.e., from $\partial\Omega_{\beta\alpha}$ to $\partial\Omega_{\alpha\beta}$) are illustrated in Fig. 2.18. The starting and ending points for mappings P_α in Ω_α are (x_i, V, t_i) and (x_{i+1}, V, t_{i+1}), respectively. The stick mapping is P_0. Define the switching planes as

$$
\begin{aligned}
\Xi^0 &= \left\{(x_i, \Omega t_i) \mid \dot{x}_i(t_i) = V\right\}, \\
\Xi^1 &= \left\{(x_i, \Omega t_i) \mid \dot{x}_i(t_i) = V^+\right\}, \\
\Xi^2 &= \left\{(x_i, \Omega t_i) \mid \dot{x}_i(t_i) = V^-\right\}
\end{aligned}
\tag{2.81}
$$

where $V^- = \lim_{\delta\to 0}(V - \delta)$ and $V^+ = \lim_{\delta\to 0}(V + \delta)$ for arbitrarily small $\delta > 0$. Therefore,

$$P_1 : \Xi^1 \to \Xi^1, \quad P_2 : \Xi^2 \to \Xi^2, \quad P_0 : \Xi^0 \to \Xi^0. \tag{2.82}$$

From the foregoing two equations, we have

$$\begin{aligned} &P_0 : (x_i, V, t_i) \to (x_{i+1}, V, t_{i+1}),\\ &P_1 : (x_i, V^+, t_i) \to (x_{i+1}, V^+, t_{i+1}),\\ &P_2 : (x_i, V^-, t_i) \to (x_{i+1}, V^-, t_{i+1}). \end{aligned} \tag{2.83}$$

The governing equations for P_0 and $\alpha \in \{1, 2\}$ are

$$\begin{aligned} &-x_{i+1} + V(t_{i+1} - t_i) + x_i = 0,\\ &2d_\alpha V + c_\alpha\big[V(t_{i+1} - t_i) + x_i\big] - A_0 \cos \Omega t_{i+1} + b_\alpha = 0. \end{aligned} \tag{2.84}$$

The mapping P_0 described the starting and ending of the stick motion, the disappearance of stick motion requires $F_\alpha(\mathbf{x}_{i+1}, \Omega t_{i+1}) = 0$. This chapter will not use this mapping to discuss the grazing flow, which is presented herein as a generic mapping. For sliding motions, this mapping will be used, and such a discussion is arranged in Luo and Gegg (2006a, 2006b). From this problem, the two domains Ω_α ($\alpha = 1$ or 2) are unbounded. The flows of the dynamical systems on the corresponding domains should be bounded from Assumptions (A1)–(A3) in nonsmooth dynamical systems. Therefore, for nonstick motion, there are three possible stable motions in the two domains Ω_α ($\alpha \in \{1, 2\}$), the governing equations of mapping P_α ($\alpha \in \{1, 2\}$) are obtained from the displacement and velocity responses for the three cases of motions in the Appendix A. Therefore, the governing equations of mapping P_α ($\alpha \in \{0, 1, 2\}$) can be expressed by

$$\begin{aligned} &f_1^{(\alpha)}(x_i, \Omega t_i, x_{i+1}, \Omega t_{i+1}) = 0,\\ &f_2^{(\alpha)}(x_i, \Omega t_i, x_{i+1}, \Omega t_{i+1}) = 0. \end{aligned} \tag{2.85}$$

If the grazing for the two nonstick mappings occurs at the final state (x_{i+1}, V, t_{i+1}), from Eq. (2.78), the grazing conditions based on mappings are obtained, i.e.,

$$\begin{aligned} &F_\alpha(x_{i+1}, V, \Omega t_{i+1}) = 0,\\ &\nabla F_\alpha(\mathbf{x}_{i+1}, \Omega t_{i+1}) \cdot \mathbf{F}_\alpha^{(\alpha)}(t_{i+1})\\ &\quad + \frac{\partial F_\alpha(\mathbf{x}_{i+1}, \Omega t_{i+1})}{\partial t} \begin{cases} > 0 & \text{for } \alpha = 1,\\ < 0 & \text{for } \alpha = 2. \end{cases} \end{aligned} \tag{2.86}$$

With Eq. (2.71), the grazing condition becomes

$$
\begin{aligned}
&A_0 \cos \Omega t_{i+1} - b_\alpha - 2d_\alpha V - c_\alpha x_{i+1} = 0,\\
&-c_\alpha V - A_0 \Omega \sin \Omega t_{i+1} \begin{cases} > 0 & \text{for } \alpha = 1,\\ < 0 & \text{for } \alpha = 2. \end{cases}
\end{aligned}
\tag{2.87}
$$

The grazing conditions for the two nonstick mappings can be illustrated with varying parameters. The grazing conditions in Eq. (2.87) are given through the forces. Hence, both the initial and final switching sets of the two nonstick mappings will vary with system parameters. Because the grazing characteristics of the two nonstick mappings are different, illustrations of grazing conditions for the two mappings will be separated. The grazing conditions are computed through Eqs. (2.85) and (2.87). Three equations plus an inequality with four unknowns require to fix one unknown. In illustrations, the initial displacement of mapping P_α ($\alpha \in \{1, 2\}$) will be fixed to specific values. Therefore, three equations with three unknowns will give the grazing conditions. Namely, the initial switching phase, the final switching phase and displacement of mapping P_α ($\alpha = 1, 2$) will be determined by Eqs. (2.85) and (2.87). To ensure the initial switching sets to be passable, from Luo (2005a, 2005b, 2006), the initial switching sets of mapping P_α ($\alpha \in \{1, 2\}$) should satisfy the following condition as in Luo and Gegg (2006a):

$$
\begin{aligned}
&F_1(x_i, V^+, \Omega t_i) < 0 \quad \text{and} \quad F_2(x_i, V^-, \Omega t_i) < 0 \quad \text{for } \Omega_1 \to \Omega_2,\\
&F_1(x_i, V^+, \Omega t_i) > 0 \quad \text{and} \quad F_2(x_i, V^-, \Omega t_i) > 0 \quad \text{for } \Omega_2 \to \Omega_1.
\end{aligned}
\tag{2.88}
$$

To make sure motions relative to mappings P_α ($\alpha = 1, 2$) exist, the initial switching force product $F_1 \times F_2$ at the boundary should be nonnegative. The comprehensive discussion of the foregoing condition can be referred to Luo and Gegg (2006b). The condition of Eq. (2.88) guarantees the flow relative to the initial switching sets of mapping P_α ($\alpha \in \{1, 2\}$) is passable on the discontinuous boundary (i.e., $y_i = V$). The force product for the initial switching sets is also illustrated to ensure the nonstick mapping exists. The force conditions for the final switching sets of mapping P_α ($\alpha \in \{1, 2\}$) is presented in Eq. (2.76). However, the equivalent grazing conditions based on Eq. (2.78) give the inequality condition in Eq. (2.87), which is already embedded in the program for computation of the grazing. Therefore, such a force product of the final switching sets of the two mappings will not be presented. From the inequality of Eq. (2.87), the critical value for $\mathrm{mod}(\Omega t_{i+1}, 2\pi)$ is introduced through

$$
\Theta_\alpha^{\mathrm{cr}} = \arcsin\left(-\frac{c_\alpha V}{A_0 \Omega}\right)
\tag{2.89}
$$

where the superscript "cr" represents a critical value relative to grazing and $\alpha \in \{1, 2\}$. From the second equation of Eq. (2.87), the final switching phase for mapping P_1 has the following six cases:

$$\begin{aligned}
&\operatorname{mod}(\Omega t_{i+1}, 2\pi) \in \left(\pi + \left|\Theta_1^{\mathrm{cr}}\right|, 2\pi - \left|\Theta_1^{\mathrm{cr}}\right|\right) \subset (\pi, 2\pi),\\
&\quad \text{for } V > 0 \text{ and } A_0\Omega > c_1 V;\\
&\operatorname{mod}(\Omega t_{i+1}, 2\pi) \in \left(\pi - \Theta_1^{\mathrm{cr}}, 2\pi\right] \cup \left[0, \Theta_1^{\mathrm{cr}}\right),\\
&\quad \text{for } V < 0 \text{ and } A_0\Omega > c_1 |V|;\\
&\operatorname{mod}(\Omega t_{i+1}, 2\pi) \in (\pi, 2\pi), \quad \text{for } V = 0;
\end{aligned} \tag{2.90}$$

and

$$\begin{aligned}
&\operatorname{mod}(\Omega t_{i+1}, 2\pi) \in \varnothing, && \text{for } V > 0 \text{ and } A_0\Omega \leqslant c_1 V;\\
&\operatorname{mod}(\Omega t_{i+1}, 2\pi) \in [0, 2\pi], && \text{for } V < 0 \text{ and } A_0\Omega < c_1 |V|;\\
&\operatorname{mod}(\Omega t_{i+1}, 2\pi) \in [0, 2\pi]/\{\pi/2\}, && \text{for } V < 0 \text{ and } A_0\Omega = c_1 |V|.
\end{aligned} \tag{2.91}$$

From the first equation of Eq. (2.87), the final displacement is bounded by

$$-A_0 \leqslant b_\alpha + 2d_\alpha V + c_\alpha x_{i+1} \leqslant A_0. \tag{2.92}$$

The spring and damper parameters ($d_1 = 1, d_2 = 0, c_1 = c_2 = 30$) are fixed as constants and the external parameters will vary. Consider the grazing variation of mapping P_1 with the belt speed V for the specified parameters ($\Omega = 1, A_0 = 20, b_1 = -b_2 = 3$). When the initial displacements $x_i = \{-1, -2, \dots, -9\}$ are specified, the initial switching phase and force products, and the final switch phase and displacement versus the belt speed are illustrated in Fig. 2.19. The initial switching phase modulus (i.e., $\operatorname{mod}(\Omega t_i, 2\pi)$) is distributed in the interval $[0, 2\pi]$. With increasing both negative, initial switching displacements and negative belt speeds, the initial switching phase will drop in the interval $[\pi, 2\pi]$. The initial switching force product on the discontinuous boundary in Fig. 2.19(b) is always positive, which is required in Eq. (2.88). For $x_i = -1$, the minimum value of the initial switching force product is close to zero. In an alike way, with increasing both negative, initial displacements and negative belt speeds, the initial force product becomes larger and larger. Such a result of the final switching phase in Eqs. (2.90) and (2.91) is confirmed by the illustration in Fig. 2.19(c). For $c_1 V > A_0\Omega$, the grazing will always occur as long as the solutions given by Eqs. (2.90) and (2.91) exist in the domain Ω_1. The final switching displacement becomes positive and large with increasing the initial switching displacement, as shown in Fig. 2.19(d). The final displacement is bounded through Eq. (2.92), which is depicted through the dark, dashed lines. Most of the grazing of mapping P_1 appears for negative belt speed. The final displacement is tangential to the upper bounded line.

Consider the grazing varying with the friction force b for mapping P_1 with $\Omega = 10, V = 1, A_0 = 20, b_1 = -b_2 = b$. The grazing condition is illustrated in Fig. 2.20 for $x_i = \{-1, -2, \dots, -5\}$ and $b \in [0, 100]$. The sufficient condition for grazing in the second equation of Eq. (2.90) is independent of the

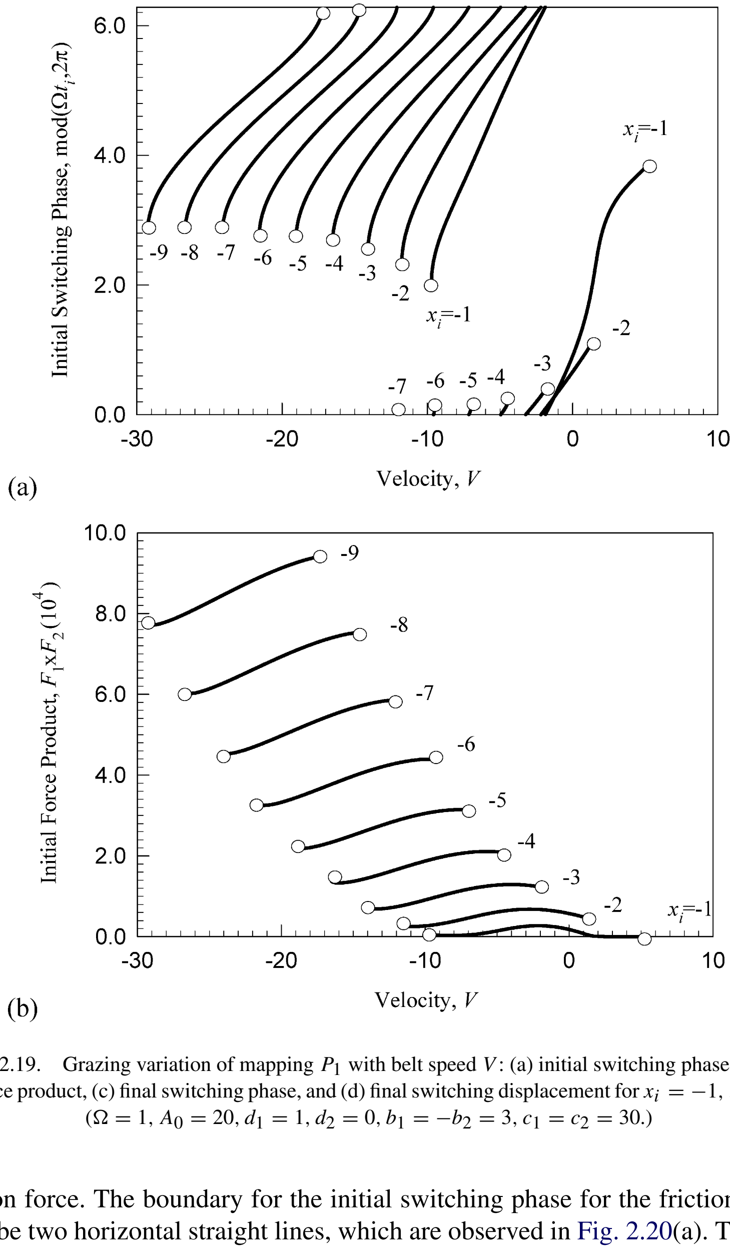
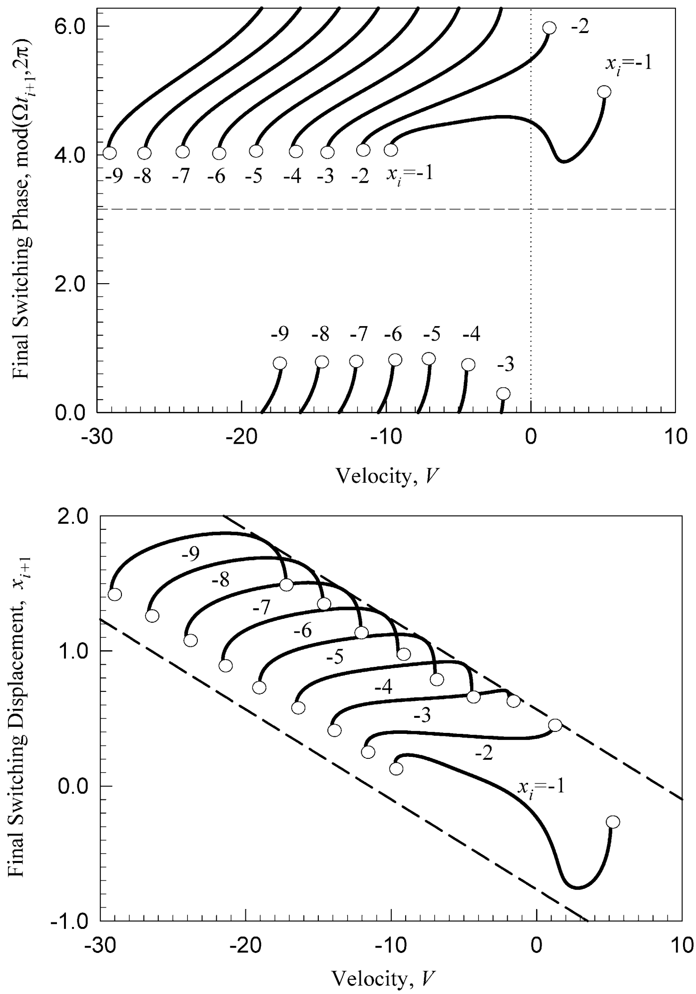


Figure 2.19. Grazing variation of mapping P_1 with belt speed V: (a) initial switching phase, (b) initial force product, (c) final switching phase, and (d) final switching displacement for $x_i = -1, \ldots, -9$. ($\Omega = 1$, $A_0 = 20$, $d_1 = 1$, $d_2 = 0$, $b_1 = -b_2 = 3$, $c_1 = c_2 = 30$.)

friction force. The boundary for the initial switching phase for the friction force must be two horizontal straight lines, which are observed in Fig. 2.20(a). The initial switching force product is presented in Fig. 2.20(b), and the appearance and disappearance boundaries of the grazing are two straight lines. Because the other parameters are given, the final switching phase should lie in $\text{mod}(\Omega t_{i+1}, 2\pi) \in$

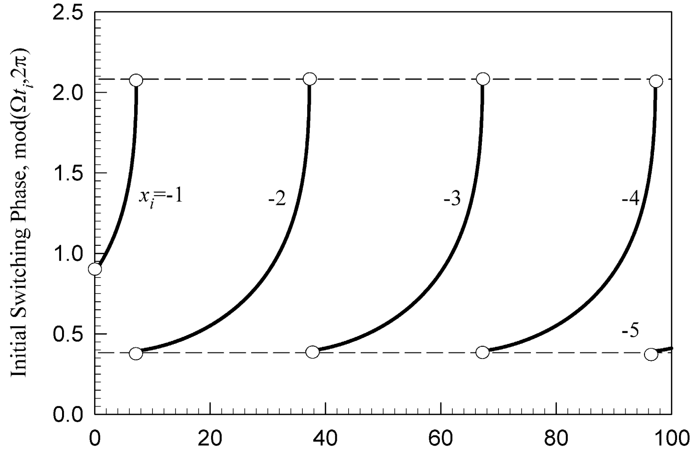


(c)

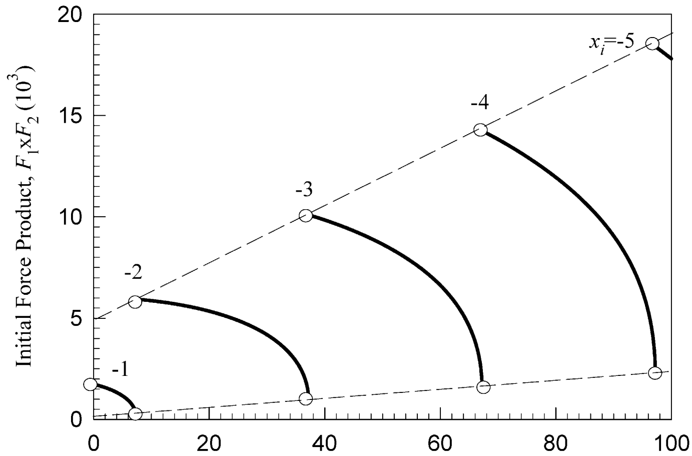
(d)

Figure 2.19. (*continued*)

[3.29, 6.13]. When the final switching phase for the grazing disappearance in Fig. 2.20(c) reaches the upper critical boundary (i.e., $\text{mod}(\Omega t_{i+1}, 2\pi) \approx 6.13$), at this time, the initial force product is not zero. In Fig. 2.20(c), the lower boundary of the final switching phase does not reach the critical boundary because there is no solution for Eqs. (2.90) and (2.91) in domain Ω_1. The final switching displacement is bounded in two parallel straight lines in Fig. 2.20(d). The grazing appearance and disappearance of mapping P_1 for the final switching displacement are linear to the friction force, which is given by Eq. (2.92). The dashed



(a)



(b)

Figure 2.20. Grazing variation of mapping P_1 with friction force b: (a) initial switching phase, (b) initial force product, (c) final switching phase, and (d) final switching displacement for $x_i = -1, -2, \ldots, -5$. ($\Omega = 10$, $A_0 = 20$, $V = 1$, $d_1 = 1$, $d_2 = 0$, $b_1 = -b_2 = b$, $c_1 = c_2 = 30$.)

dark lines give the boundary. The final displacement arrives to the upper boundary before the force product becomes zero. The lower boundary is controlled by the sufficient conditions.

Consider the grazing variation of mapping P_1 with excitation amplitude A_0 as the parameters $\Omega = 8, V = 1, b_1 = -b_2 = 3$ are used. Illustrations of the

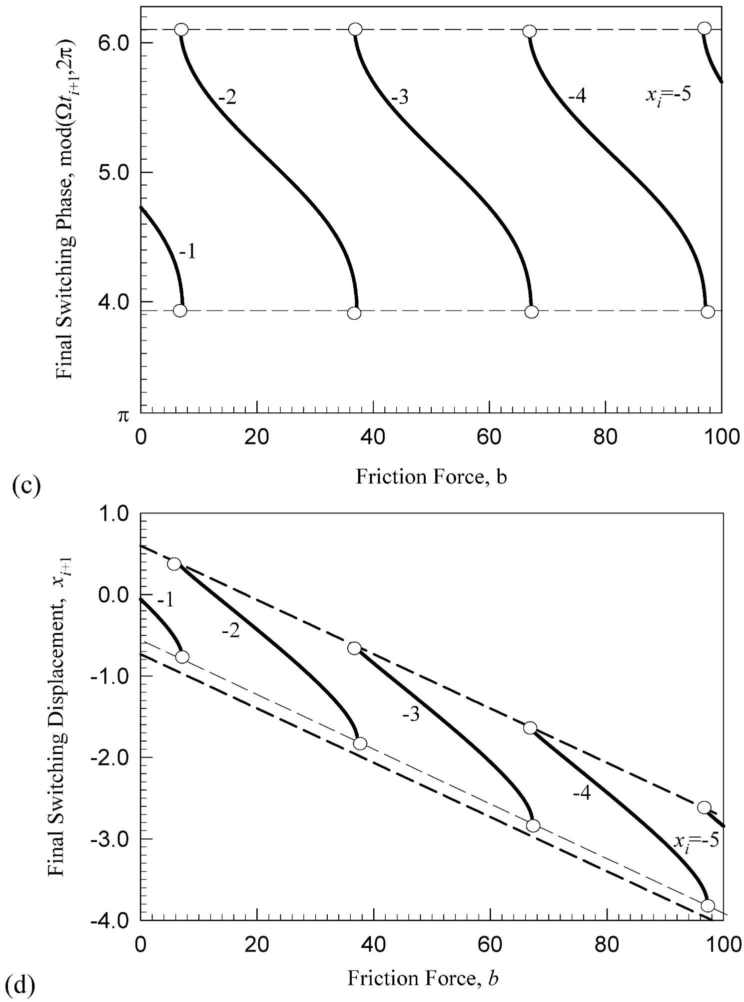


Figure 2.20. (*continued*)

grazing condition is presented in Fig. 2.21 for $x_i = -1, -2, \ldots, -6$. Owing to $V = 1 > 0$, the sufficient condition for grazing requires $A_0 > 3.75$, and the final switching phase will be in the interval $(\pi, 2\pi)$ in Eq. (2.91). The corresponding initial force product is positive. The grazing band of the excitation amplitude becomes large with increasing initial switching displacements. The upper bounded line in the final displacement is to control the grazing existence. In Fig. 2.22, the grazing varying with excitation frequency Ω for mapping P_1 is

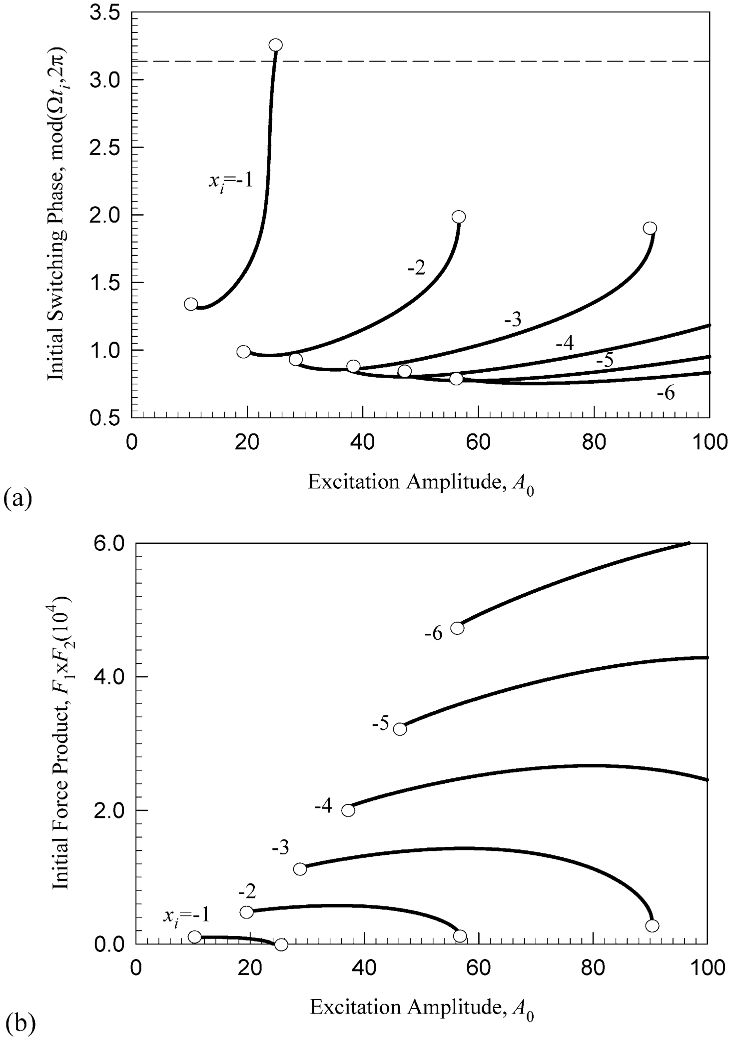


Figure 2.21. Grazing variation of mapping P_1 with excitation amplitude A_0: (a) initial switching phase, (b) initial force product, (c) final switching phase, and (d) final switching displacement for $x_i = -1, -2, \ldots, -6$. ($\Omega = 8$, $V = 1$, $d_1 = 1$, $d_2 = 0$, $b_1 = -b_2 = 3$, $c_1 = c_2 = 30$.)

shown for $A_0 = 20$, $V = 1$, $b_1 = -b_2 = 3$ with $x_i = -0.9, -1.0, \ldots, -2$. From Fig. 2.22(b), the minimum of the initial force product approaches zero for $x_i = -0.9$. For $x_i > -0.9$, it is very difficult to find the wide spectrum of excitation frequency for grazing solutions obtained from Eqs. (2.90) and (2.91). The

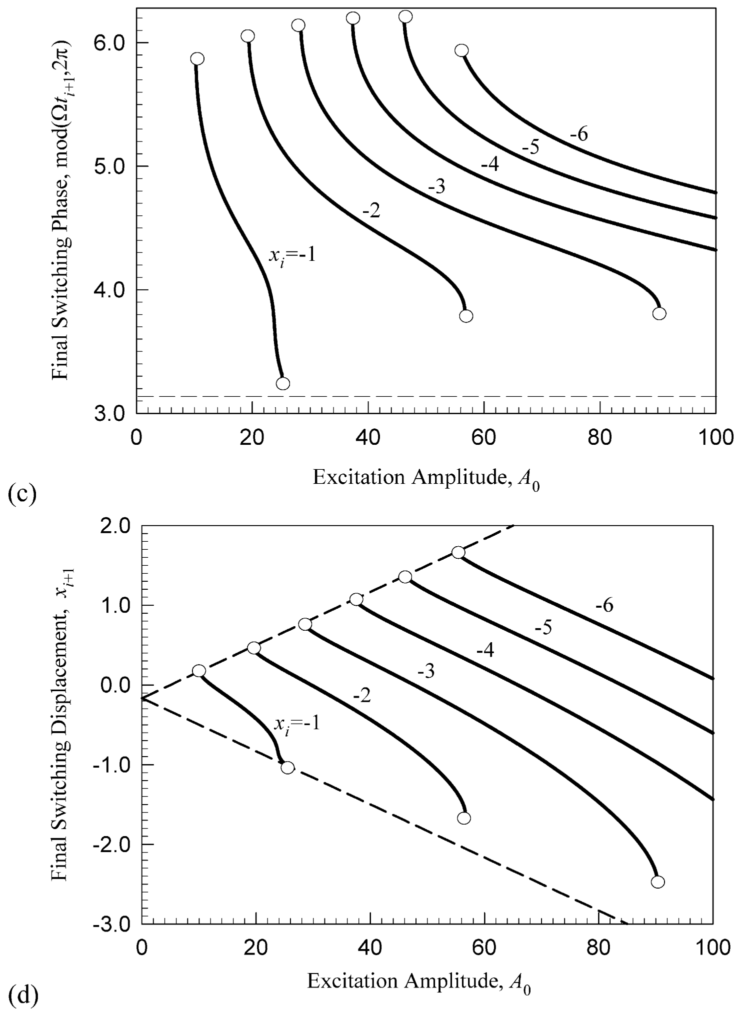


Figure 2.21. (*continued*)

initial switching phase for $x_i \in [-1.1, -0.9]$ takes the entire range of $[0, 2\pi]$. However, the final switching phase and displacement are in the narrow band for large excitation frequency. For $x_i < -1.25$, the grazing for mapping P_1 exists only in the small range of excitation frequency because the grazing conditions catch the upper bounded lines. However, for $x_i \in [-1.1, -0.9]$, the grazing for mapping P_1 exists almost in the large range of excitation frequency. It implies that the grazing exists in a wide spectrum of excitation for such initial displacements.

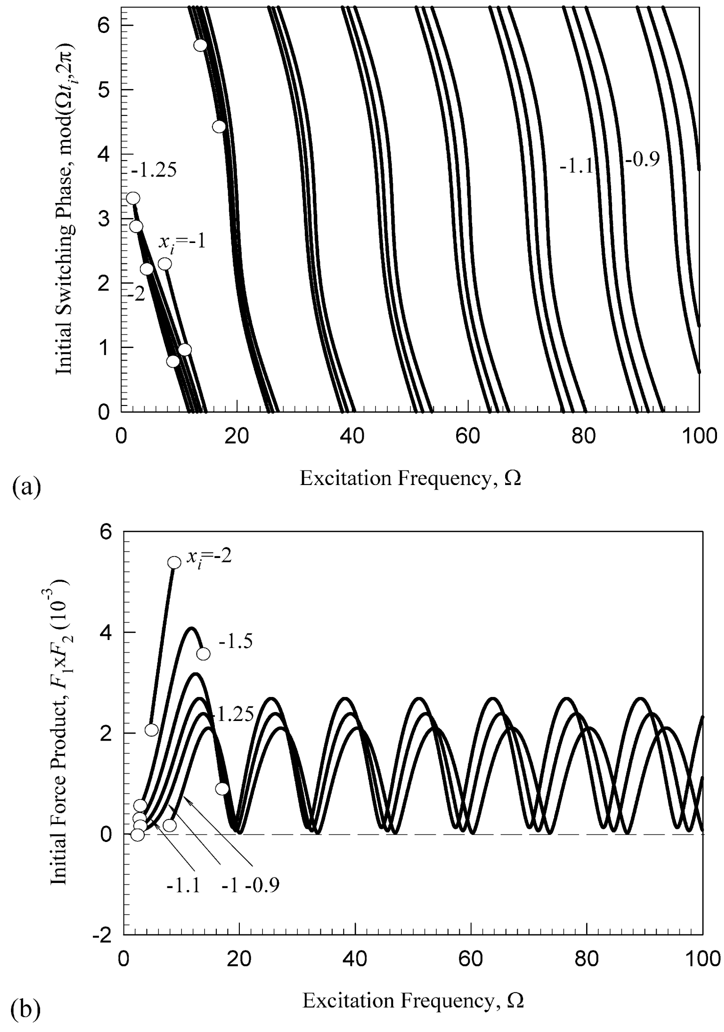


Figure 2.22. Grazing variation of mapping P_1 with excitation frequency Ω: (a) initial switching phase, (b) initial force product, (c) final switching phase, and (d) final switching displacement for $x_i = -0.9, -1.0, \ldots, -2$. ($A_0 = 20$, $V = 1$, $d_1 = 1$, $d_2 = 0$, $b_1 = -b_2 = 3$, $c_1 = c_2 = 30$.)

From the parameter characteristics of grazing for mapping P_1, it is observed that the grazing is affected by many parameters. In the two domains, the system parameters are different. Therefore, the parameter characteristics of grazing for mapping P_2 will be presented as follows. Similarly, from the second equation

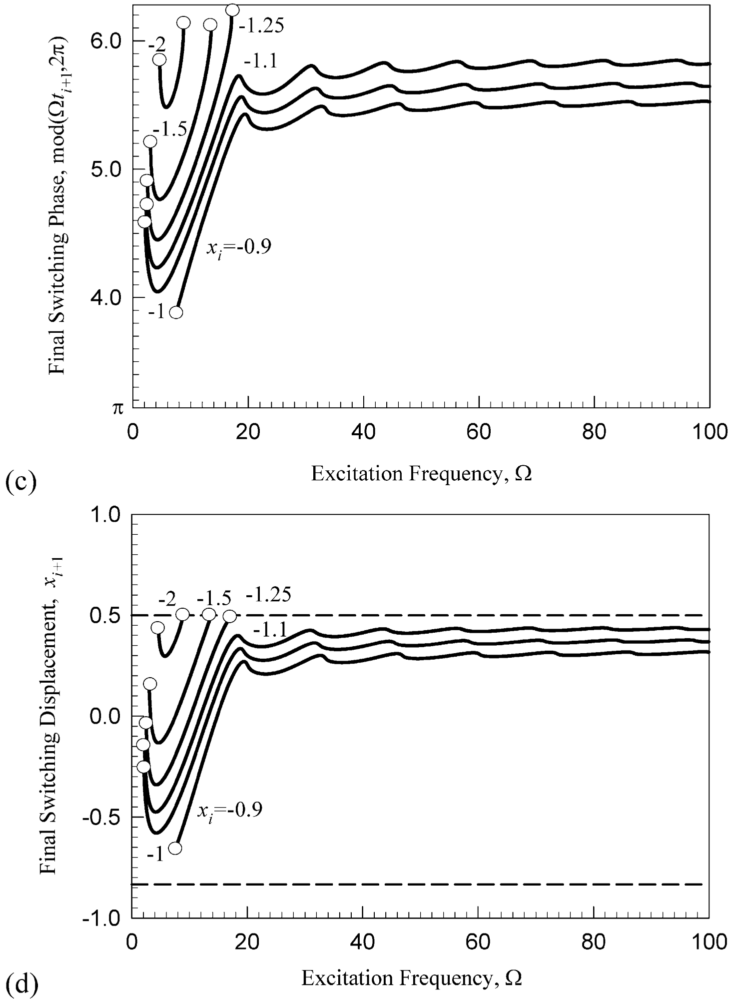


Figure 2.22. (*continued*)

of Eq. (2.87), the six cases of the final switching phase for mapping P_2 are:

$$
\begin{aligned}
&\operatorname{mod}(\Omega t_{i+1}, 2\pi) \in \left[0, \pi + \left|\Theta_2^{\mathrm{cr}}\right|\right) \cup \left(2\pi - \left|\Theta_2^{\mathrm{cr}}\right|, 2\pi\right], \\
&\quad \text{for } V > 0 \text{ and } A_0\Omega > c_2 V; \\
&\operatorname{mod}(\Omega t_{i+1}, 2\pi) \in \left(\Theta_2^{\mathrm{cr}}, \pi - \Theta_2^{\mathrm{cr}}\right) \subset (0, \pi), \\
&\quad \text{for } V < 0 \text{ and } A_0\Omega > c_2|V|; \\
&\operatorname{mod}(\Omega t_{i+1}, 2\pi) \in (0, \pi), \quad \text{for } V = 0;
\end{aligned}
\tag{2.93}
$$

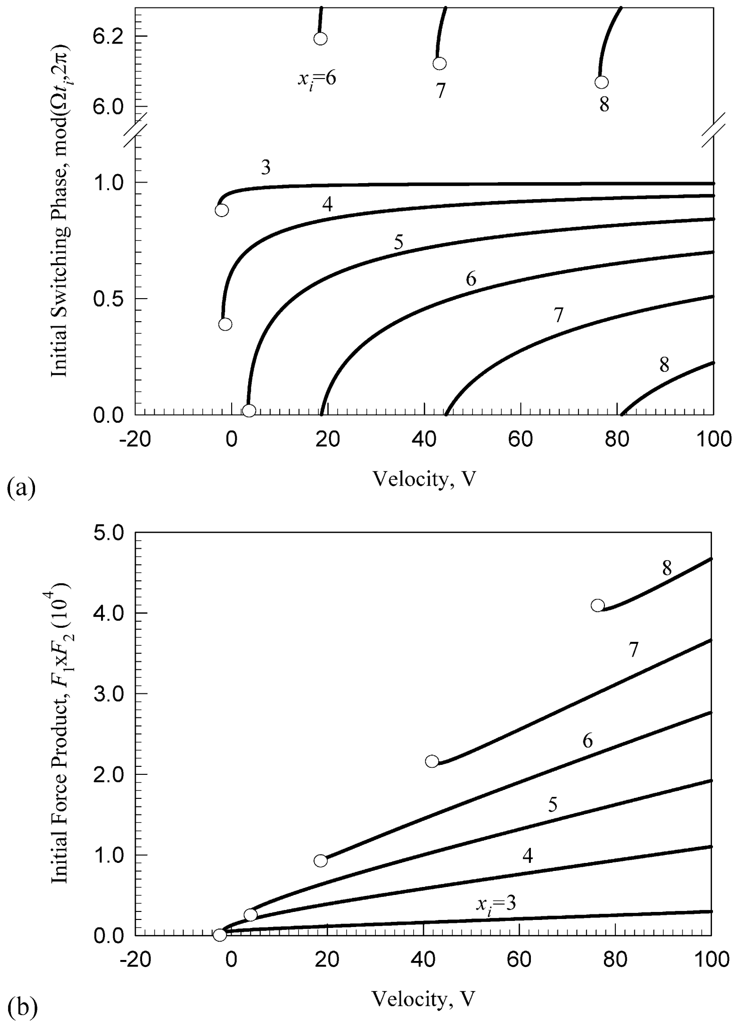


Figure 2.23. Grazing variation of mapping P_2 with belt speed V for $x_i = 3, 4, \ldots, 8$: (a) initial switching phase, (b) initial force product, (c) final switching phase, and (d) final switching displacement. ($A_0 = 90$, $\Omega = 1$, $d_1 = 1$, $d_2 = 0$, $b_1 = -b_2 = 30$, $c_1 = c_2 = 30$.)

and

$$\begin{aligned}
&\mathrm{mod}(\Omega t_{i+1}, 2\pi) \in [0, 2\pi], && \text{for } V > 0 \text{ and } A_0\Omega < c_2 V;\\
&\mathrm{mod}(\Omega t_{i+1}, 2\pi) \in [0, 2\pi]/\{3\pi/2\}, && \text{for } V > 0 \text{ and } A_0\Omega = c_2 V;\\
&\mathrm{mod}(\Omega t_{i+1}, 2\pi) \in \varnothing, && \text{for } V < 0 \text{ and } A_0\Omega < c_2 |V|.
\end{aligned} \tag{2.94}$$

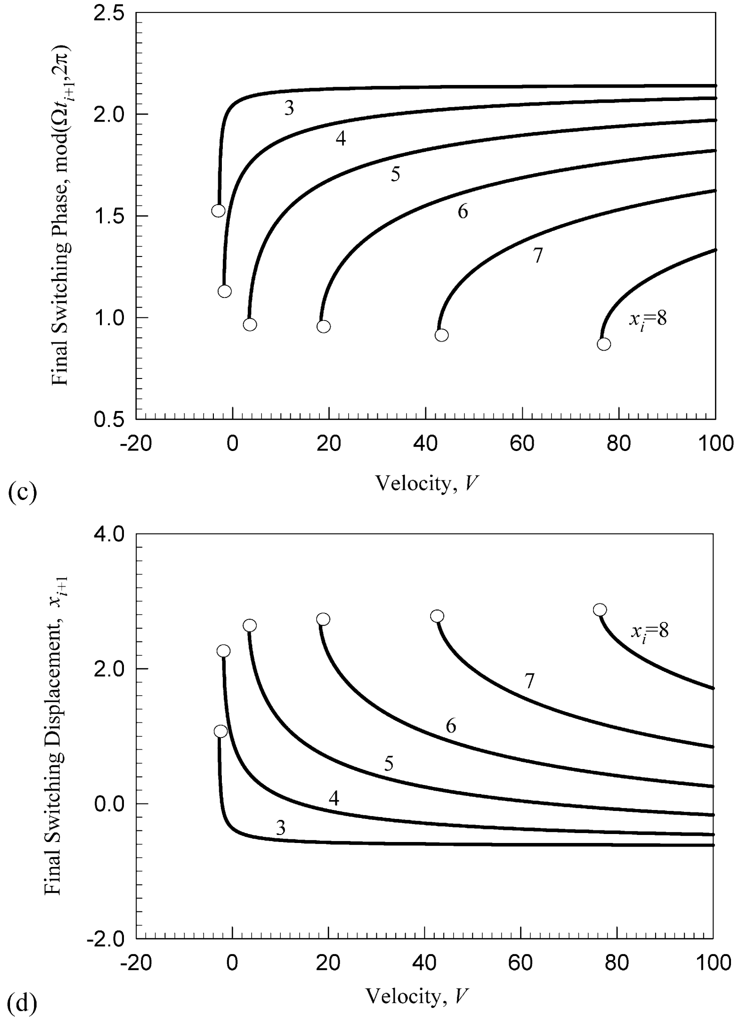


Figure 2.23. (*continued*)

Consider the same system parameters ($d_1 = 1, d_2 = 0, c_1 = c_2 = 30$). The grazing varying with the belt speed V for mapping P_2 are presented in Fig. 2.23 for $x_i = 3, 4, \ldots, 8$ under the other parameters $A_0 = 90, \Omega = 1, b_1 = -b_2 = 30$. From the aforementioned parameters, the final switching phase for $V > 3$ will be in the entire interval of $[0, 2\pi]$ as long as the grazing solution of Eqs. (2.85) and (2.87) exists for the domain Ω_2. In Fig. 2.23(a), the initial switching phase lies in $(0, \pi/2) \cup (3\pi/2, 2\pi)$. With increasing the initial switching phase, the belt

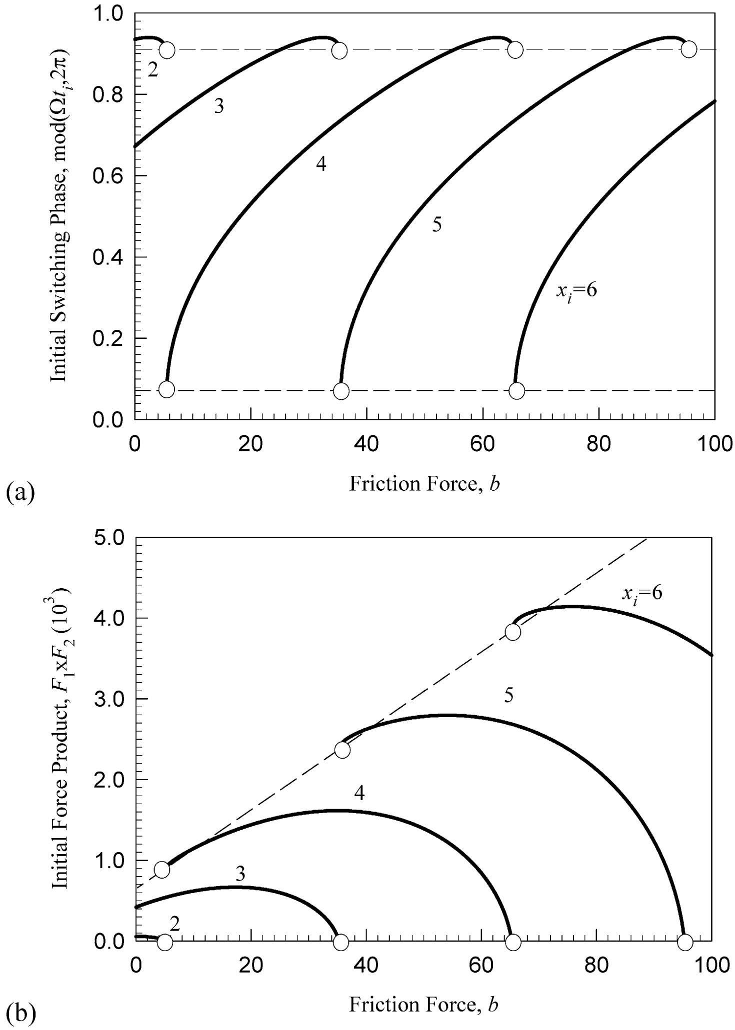
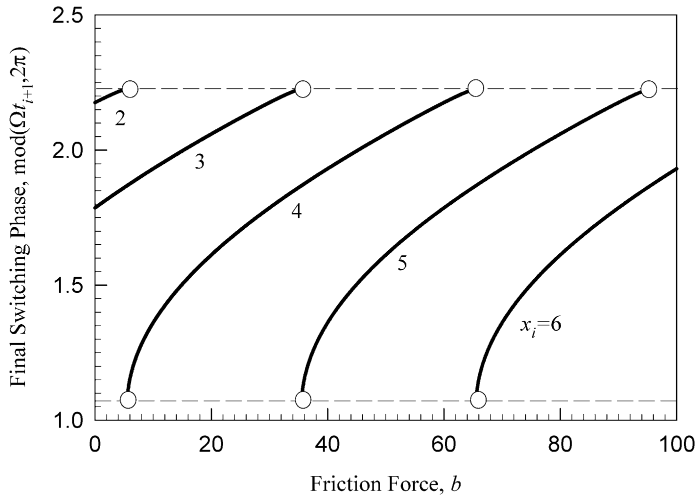
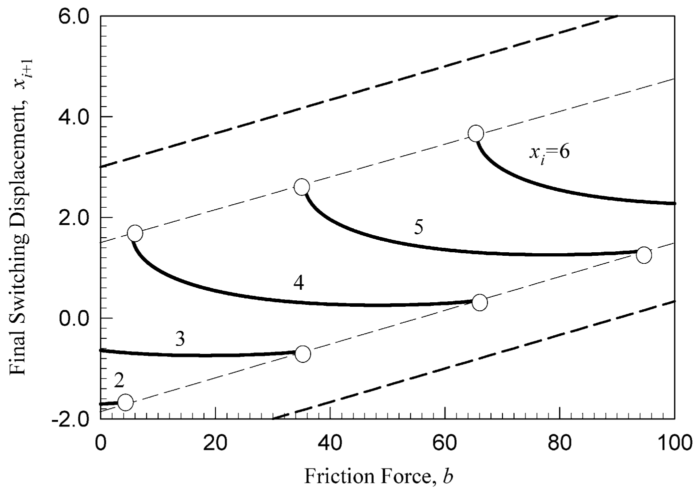


Figure 2.24. Grazing varying with friction force b for mapping P_2 with $x_i = 2, 4, \ldots, 6$: (a) initial switching phase, (b) initial force product, (c) final switching phase, and (d) final switching displacement. ($A_0 = 90$, $\Omega = 1.1$, $V = 1$, $d_1 = 1$, $d_2 = 0$, $b_1 = -b_2 = 30$, $c_1 = c_2 = 30$.)

speeds for grazing of mapping P_2 will become large. For higher belt speeds, the positive, initial force product will be obtained, as shown in Fig. 2.23(b). The final switching phase lies in the interval $(0, \pi)$ in Fig. 2.23(c). The final switching displacement in Fig. 2.23(d) is much smaller than the initial switching displacement (i.e., $x_{i+1} < x_i$). The final displacement is in the region bounded by the



(c)



(d)

Figure 2.24. (*continued*)

upper and lower boundaries by Eq. (2.92) (i.e., $x_{i+1} = -2$ and $x_{i+1} = 4$). Without damping, the upper and lower boundaries for the final displacement are independent of the velocity. The upper and lower bounded boundaries were not presented. For this case of mapping P_2, the boundaries of grazing are determined by the sufficient conditions. The grazing of mapping P_2 mostly occurs in the positive range of the belt speed. However, the grazing of mapping P_1 mostly exists in the negative range of the belt speed. In Fig. 2.24, the grazing vary-

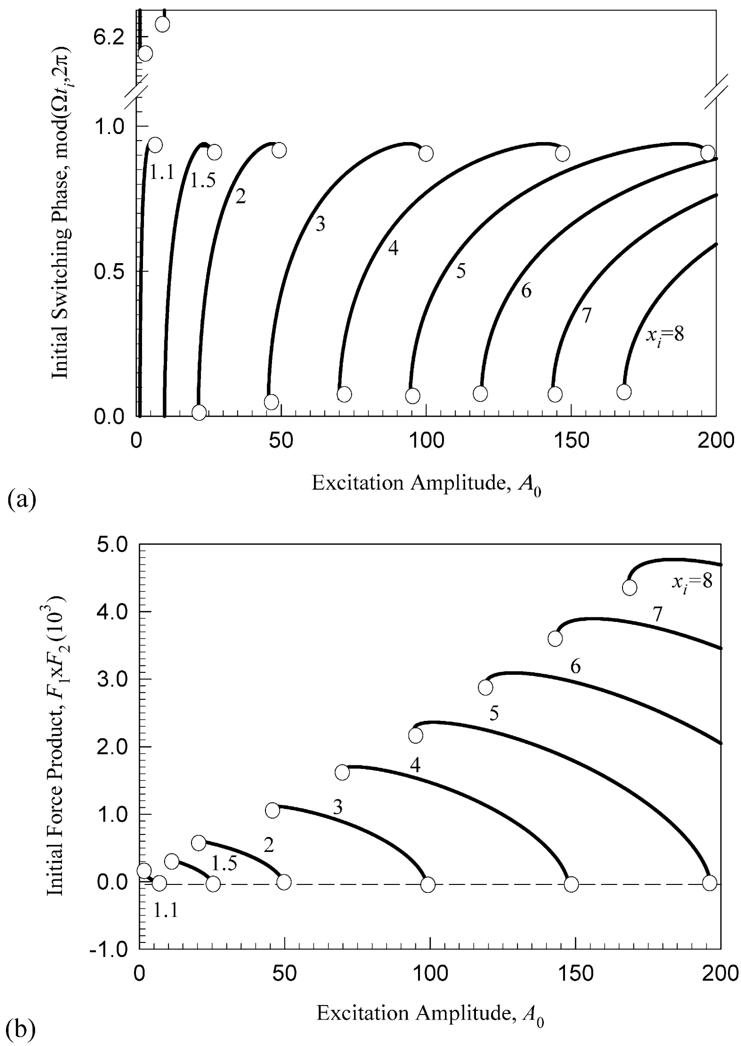
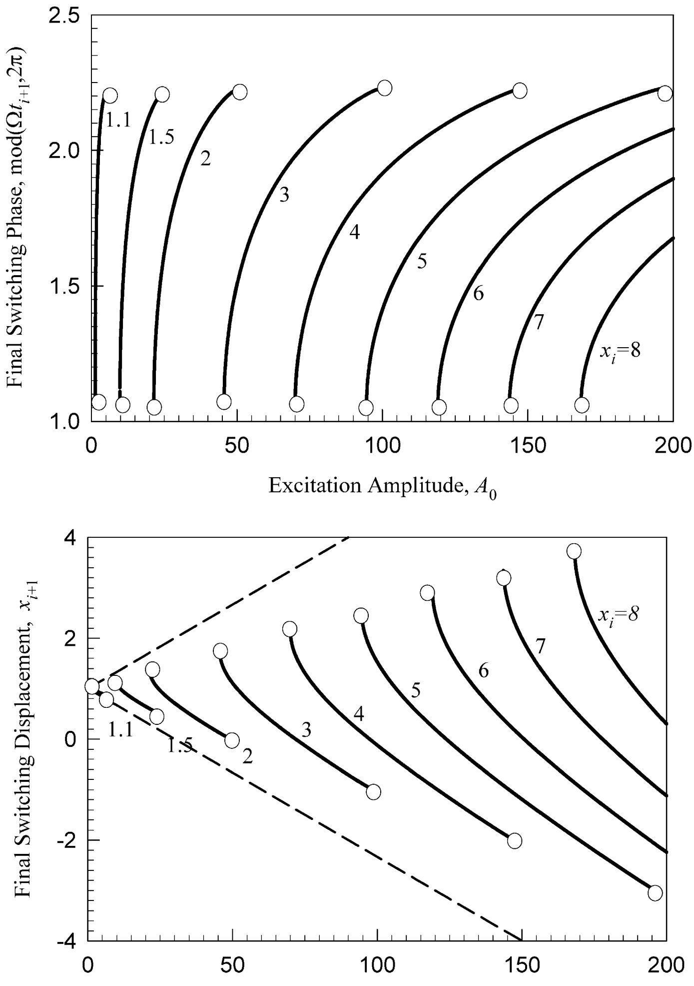


Figure 2.25. Grazing varying with excitation amplitude A_0 for mapping P_2 with $x_i = 1.1, 1.5, \ldots, 8$: (a) initial switching phase, (b) initial force product, (c) final switching phase, and (d) final switching displacement. ($\Omega = 1.1$, $V = 1$, $d_1 = 1$, $d_2 = 0$, $b_1 = -b_2 = 30$, $c_1 = c_2 = 30$.)

ing with friction force for mapping P_2 is illustrated for $x_i = 2, 4, \ldots, 6$ with $A_0 = 90$, $\Omega = 1.1$, $V = 1$, $b_1 = -b_2 = b$. The final switching phase should be in the interval $(0, 3.44946) \cup (5.96731, 2\pi)$. However, compared to the grazing of mapping P_1, the initial switching force product becomes zero before the final

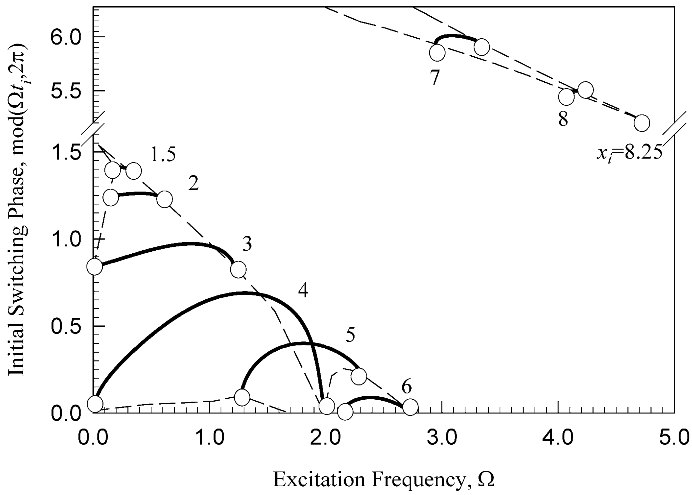


(c)

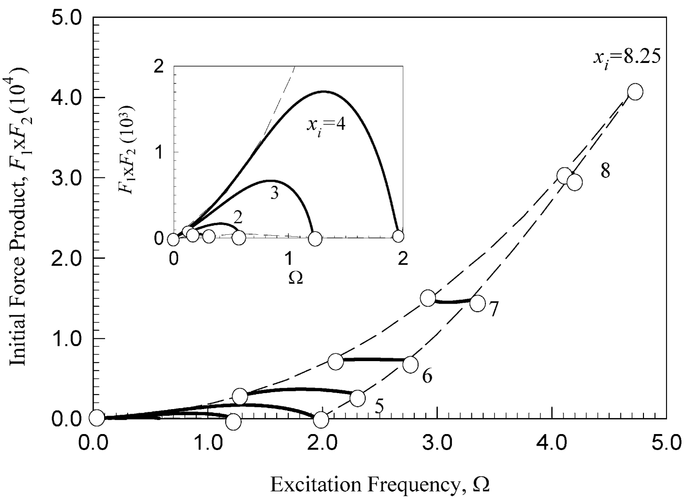
(d)

Figure 2.25. (*continued*)

switching phase reaches the critical values, as illustrated in Fig. 2.24(b). Again, the two switching phases for appearance and disappearance of grazing mappings are independent of the frictional force, as observed in Figs. 2.24(a) and (c). Such a characteristic is determined by the grazing sufficient condition. The appearance and disappearance for the final switching displacement are linear to the frictional force, as shown in Fig. 2.24(d). The final displacement lies in the region between



(a)



(b)

Figure 2.26. Grazing varying with excitation frequency Ω for mapping P_2 with $x_i = 1.5, 2, \ldots, 8.25$: (a) initial switching phase, (b) initial force product, (c) final switching phase, and (d) final switching displacement. ($A_0 = 90, V = 1, d_1 = 1, d_2 = 0, b_1 = -b_2 = 30$, $c_1 = c_2 = 30$.)

the upper and lower bounded boundary of necessary conditions. The grazing varying with excitation amplitude for mapping P_2 is also illustrated in Fig. 2.25 for $x_i = 1.1, 1.5, \ldots, 8$ and $\Omega = 1.1$, $V = 1$, $b_1 = -b_2 = 30$. The upper boundary for grazing disappearance is generated by the zero of the initial force product.

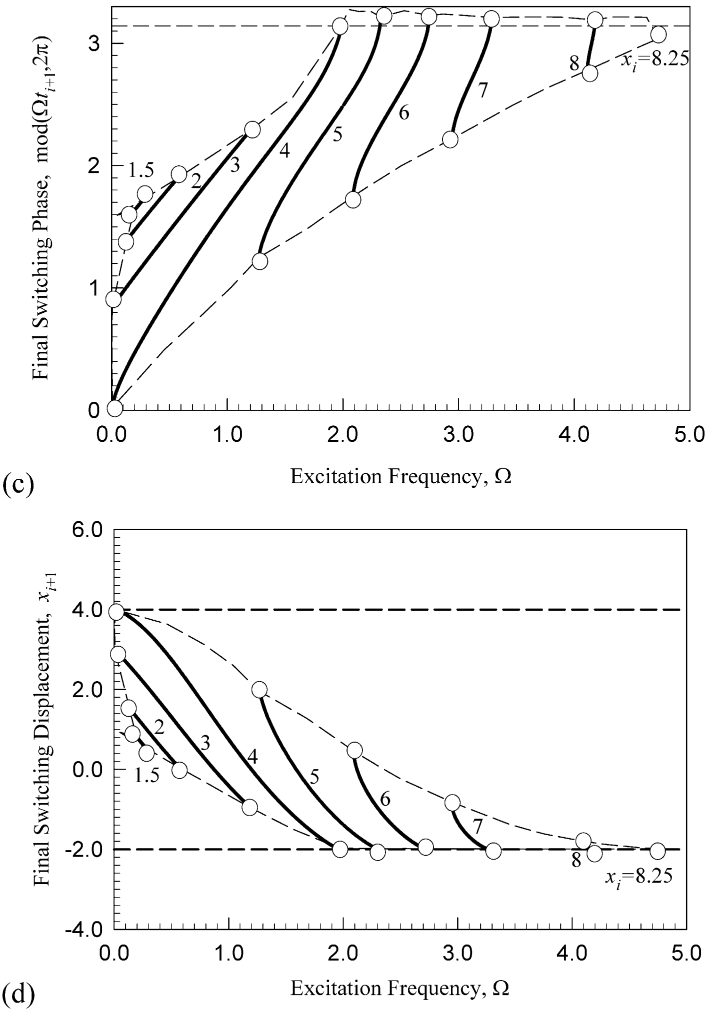


Figure 2.26. (*continued*)

The final switching displacement is in the region between the upper and lower boundaries of the necessary grazing conditions. It seems that the grazing is determined by its sufficient condition and solution existence. Finally, the grazing varying with excitation frequency Ω for mapping P_2 is presented in Fig. 2.26 for $x_i = 1.5, 2, \ldots, 8.25$ and $A_0 = 90$, $V = 1$, $b_1 = -b_2 = 30$. The closed dashed line is the boundary for the grazing of mapping P_2. No more grazing of mapping P_2 will exist for $x_i > 8.25$. The initial switching phase and force product are pre-

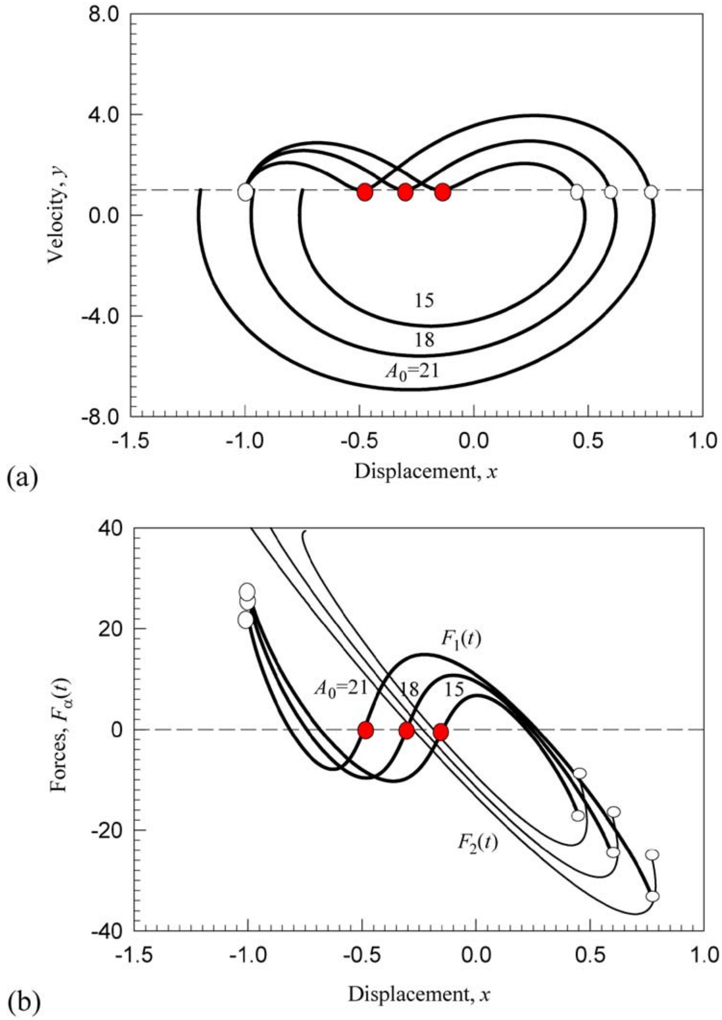


Figure 2.27. Grazing motion of mapping P_1 for $A_0 = 15, 18, 21$: (a) phase trajectory, (b) forces distribution along displacement, (c) velocity–time history, and (d) forces distribution on velocity. ($\Omega = 8, V = 1, d_1 = 1, d_2 = 0, b_1 = -b_2 = 3, c_1 = c_2 = 30$.) The initial conditions are $(x_i, y_i) = (-1.0, 1.0)$ and $\Omega t_i \approx 1.3617, 1.6958, 1.4830$, respectively.

sented in Figs. 2.26(a) and (b), respectively. For small excitation frequencies, the disappearance for grazing mapping P_2 is caused by the zero of the initial force product. For large excitation frequencies, the grazing disappearance is because the sufficient condition in Eq. (2.87) cannot be satisfied.

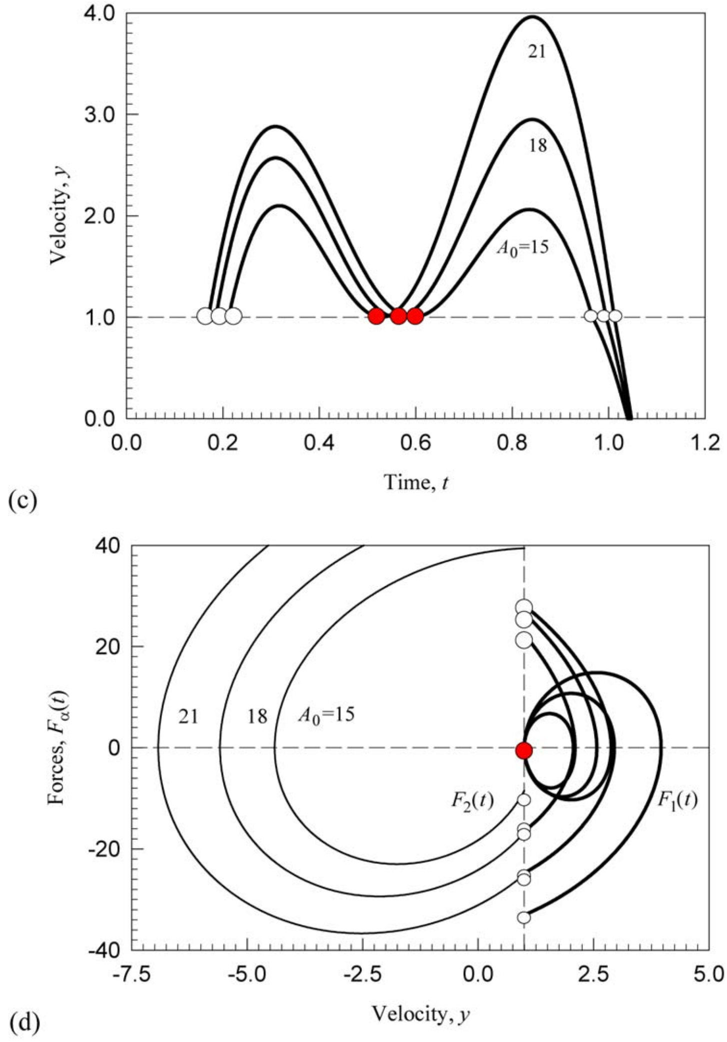


Figure 2.27. (*continued*)

To verify the analytical prediction of the grazing motions, the motion responses of the oscillator will be demonstrated through time-history responses and trajectories in phase plane. The grazing strongly depends on the force responses in this discontinuous dynamical system. The force responses will be presented to illustrate the force criteria for the grazing motions in such a friction-induced oscillator. The starting and grazing points of mapping P_α ($\alpha \in \{1, 2\}$) are represented by the

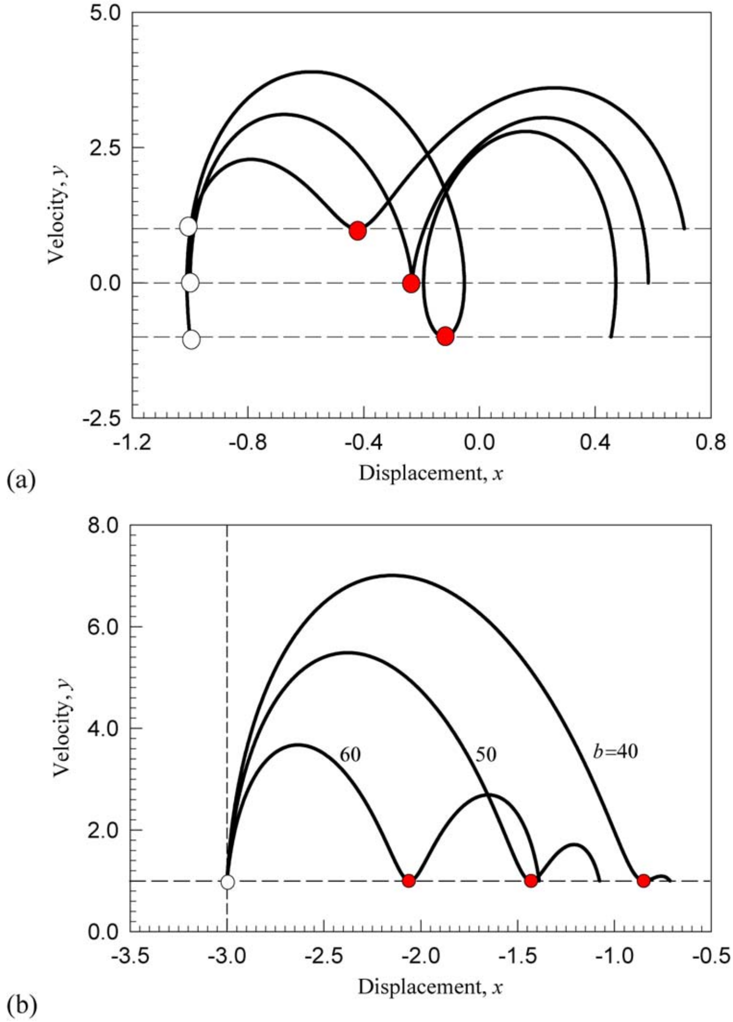


Figure 2.28. Grazing phase trajectories of mapping P_1 ($A_0 = 20, \Omega = 8, d_1 = 1, d_2 = 0, b_1 = -b_2 = b, c_1 = c_2 = 30$): (a) $x_i = -1.0, \Omega t_i \approx \{0.3970, 0.9060, 1.6092\}$ for $y_i \equiv V = \{-1, 0, 1\}$ with $b = 3$, respectively. (b) $(x_i, y_i) = (-3.0, 1.0)$ with $\Omega t_i \approx \{0.3285, 0.4405, 0.7065\}$ for $b = 40, 50, 60$ and $V = 1$, respectively.

large, hollow and dark-solid circular symbols, respectively. The switching points from domain α to β ($\alpha, \beta \in \{1, 2\}, \alpha \neq \beta$) are depicted by smaller circular symbols. In Fig. 2.27, phase trajectories, forces distribution along displacement, velocity time-history and forces distribution on velocity are presented for the graz-

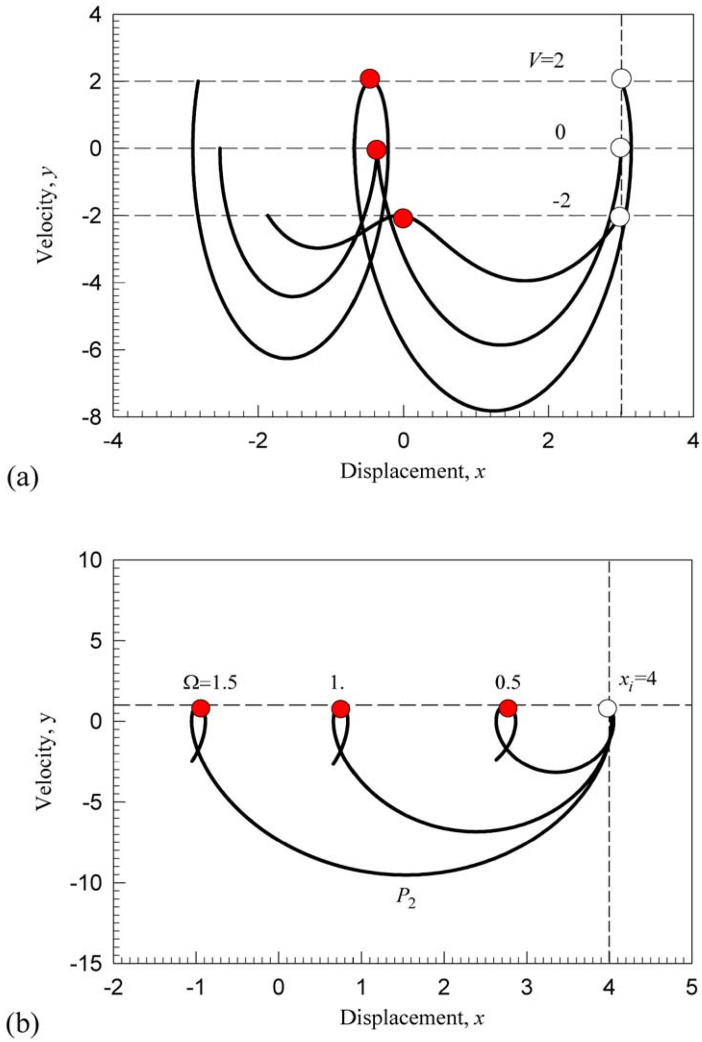


Figure 2.29. Grazing phase trajectories of mapping P_2 ($A_0 = 90, d_1 = 1, d_2 = 0, b_1 = -b_2 = 30$): (a) $x_i = 3.0$, $\Omega t_i \approx \{0.9297, 0.9551, 0.9653\}$ for $y_i \equiv V = \{-2, 0, 2\}$ and $\Omega = 1$, respectively; (b) $(x_i, y_i) = (4.0, 1.0)$, $\Omega t_i \approx \{0.4459, 0.6495, 0.6700\}$ for $\Omega = \{0.5, 1, 1.5\}$ and $V = 1$, respectively.

ing motion of mapping P_1. The parameters $\Omega = 8$, $V = 1$, $b_1 = -b_2 = 3$ plus the initial conditions $(x_i, y_i) = (-1.0, 1.0)$ and $\Omega t_i \approx 1.3617, 1.6958, 1.4830$ corresponding to $A_0 = 15, 18, 21$ are used. In phase plane, the three grazing trajectories are tangential to the discontinuous boundary (i.e., $y = V$), which are seen in Fig. 2.27(a). In Fig. 2.27(b), the thick and thin solid curves represent the forces $F_1(t)$ and $F_2(t)$, respectively. From the force distribution along displacement, the force $F_1(t)$ has a sign change from negative to positive. This indicates that the grazing conditions in Eq. (2.76) are satisfied. The forces $F_1(t)$ and $F_2(t)$ at the switching points from domain Ω_1 to Ω_2 have a jump with the same sign, which satisfies Eq. (2.77). In the velocity time-history plot, the velocity curves are tangential to the discontinuous boundary (see Fig. 2.27(c)). Finally, the forces distributions along velocity are presented in Fig. 2.27(d). The force $F_1(t)$ at the grazing points is zero. The force jump from domain Ω_1 to Ω_2 is observed as well. The phase trajectories for mapping P_1 are presented in Figs. 2.28(a) and (b). The parameters $A_0 = 20$, $\Omega = 8$, $b_1 = -b_2 = b$ are used. The initial conditions $x_i = -1.0$, $y_i = -1, 0, 1$ are adopted in Fig. 2.28(a) for $\Omega t_i \approx 0.3970, 0.9060$ and 1.6092. The initial conditions $(x_i, y_i) = (-3.0, 1.0)$ with $\Omega t_i \approx 0.3285, 0.4405, 0.7065$ are used for $b = 40, 50, 60$ and $V = 1$, respectively. Similarly, the phase trajectories of grazing motions for mapping P_2 are illustrated in Figs. 2.29(a) and (b). The two sets of initial conditions and parameters are $x_i = 3.0$, $\Omega t_i \approx 0.9297, 0.9551, 0.9652$ for $y_i = -2, 0, 2$ and $\Omega = 1$, respectively, and $(x_i, y_i) = (4.0, 1.0)$, $\Omega t_i \approx 0.4459, 0.6495, 0.6700$ for $\Omega = 0.5, 1.0, 1.5$ and $V = 1$, respectively, and the other parameters ($A_0 = 90$, $b_1 = -b_2 = 30$) are employed as well. The periodic motion with stick and nonstick of this oscillator can be found in Luo and Gegg (2006a, 2006b). The methodology can be applied to the other discontinuous systems for grazing flows.

Chapter 3

Flow Switching Bifurcations

In the previous chapter, the passability of flows between two adjacent accessible domains was discussed. The tangency of flows to the separation boundary was discussed. The sliding flows on the nonpassable boundary are important to understand the global flows in discontinuous dynamical systems. In this chapter, the sliding dynamics on the separation boundary will be discussed based on the set-valued vector field theory. From vector fields in the neighborhood of a specific separation boundary, the passability of the flow from the one domain into another one will be further discussed. The switching bifurcation conditions from the passable boundary to the nonpassable boundary will be developed. The sliding flow fragmentation on the separation boundary surface will be presented. The normal vector product function will be introduced to determine the switching bifurcation and sliding fragmentation.

3.1. Set-valued vector fields

Consider a flow $\mathbf{x}^{(\alpha)}(t)$ ($\alpha \in \{i, j\}$) in the vicinity of the boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta$, with $\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t) > 0$ in the domain Ω_α for $t \in [t_{m-\varepsilon}, t_m)$, as shown in Fig. 2.5. In the domain Ω_β ($\alpha \neq \beta$), the flow $\mathbf{x}^{(\beta)}(t)$ in the vicinity of the boundary $\partial\Omega_{ij}$ possesses the condition $\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}(t) < 0$ at the same time interval. Therefore, the normal component product of the vector fields of the two flows satisfies the condition in Eq. (2.13), i.e., $\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-})\}\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m-})\} < 0$. From Theorem 2.3, such a condition implies that the boundary $\partial\Omega_{ij}$ is nonpassable. It means that no motion exists in the normal direction of the boundary $\partial\Omega_{ij}$. Further, only the sliding motion exists on the portion of the boundary. If no sliding motion exists on the boundary, the portion of the boundary $\partial\Omega_{ij}$ is termed the *static* discontinuous boundary. For instance, an oscillator moves on the stationary surface with dry friction, and its discontinuous boundary is static. In mechanical systems, the discontinuous displacement boundary is also static. If the sliding motion exists, until one of the two flows (i.e., $\mathbf{x}^{(\gamma)}(t)$, $\gamma \in \{\alpha, \beta\}$) has

$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\gamma)}(t_{m+}) = 0$ with $\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-\varepsilon})\}\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+\varepsilon})\} = 0$, the sliding motion will vanish.

To investigate the sliding motion along the nonpassable separation surface, the sliding dynamics is very important. Even for the passable boundary, the sliding dynamics will also affect the dynamic characteristics of the output flow from the boundary. As in Fillippov (1988), consider a differential inclusion of Eq. (2.1) on the closed interval [0, 1] as

$$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}, t, \lambda), \quad \mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \Omega_i \cup \Omega_j \cup S_{ij} \tag{3.1}$$

where a set-valued vector field $\mathbf{F}(\mathbf{x}, t, \lambda)$ is convex and continuous with respect to the parameter λ on the closed interval [0, 1]. The following property holds for the convex set of the vector field:

$$\mathbf{F}(\mathbf{x}, t, \lambda) \begin{cases} = \mathbf{F}^{(\alpha)}(\mathbf{x}, t, \boldsymbol{\mu}_\alpha), & \text{for input vector filed in } \Omega_\alpha,\ \lambda = 0, \\ \in \mathbf{F}^{(0)}_{\alpha\beta}(\mathbf{x}, t), & \text{on the boundary } \partial\Omega_{\alpha\beta},\ \lambda \in (0, 1), \\ = \mathbf{F}^{(\beta)}(\mathbf{x}, t, \boldsymbol{\mu}_\beta), & \text{for output vector in } \Omega_\beta,\ \lambda = 1 \end{cases} \tag{3.2}$$

where $\mathbf{F}^{(\alpha)}$ and $\mathbf{F}^{(\beta)}$ $(\alpha, \beta \in \{i, j\},\ \alpha \neq \beta)$ represent the input and output vector fields, respectively. $\mathbf{F}^{(0)}_{\alpha\beta}(\mathbf{x}, t)$ is a vector field along the separation boundary $\partial\Omega_{\alpha\beta}$. From the convexity of the set-valued vector field, we have

$$\mathbf{F}^{(0)}_{\alpha\beta}(\mathbf{x}, t) = \lambda\mathbf{F}^{(\beta)}(\mathbf{x}, t, \boldsymbol{\mu}_\beta) + (1 - \lambda)\mathbf{F}^{(\alpha)}(\mathbf{x}, t, \boldsymbol{\mu}_\alpha). \tag{3.3}$$

The sliding motion is along the separation boundary, it indicates that the vector field is along the boundary. So $\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(0)}_{\alpha\beta} = 0$ from which we have

$$\lambda = \frac{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(\mathbf{x}, t, \boldsymbol{\mu}_\alpha)}{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot [\mathbf{F}^{(\alpha)}(\mathbf{x}, t, \boldsymbol{\mu}_\alpha) - \mathbf{F}^{(\beta)}(\mathbf{x}, t, \boldsymbol{\mu}_\beta)]}. \tag{3.4}$$

For the traveling separation boundary controlled by $\varphi_{ij}(\mathbf{x}, t)$, the total derivative and convexity of $\varphi_{ij}(\mathbf{x}, t)$ gives

$$\lambda = \frac{\frac{\partial}{\partial t}\varphi_{\alpha\beta} + \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(\mathbf{x}, t, \boldsymbol{\mu}_\alpha)}{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot [\mathbf{F}^{(\alpha)}(\mathbf{x}, t, \boldsymbol{\mu}_\alpha) - \mathbf{F}^{(\beta)}(\mathbf{x}, t, \boldsymbol{\mu}_\beta)]}. \tag{3.5}$$

The sliding motion along the separated boundary can be investigated as a continuous dynamical system through

$$\dot{\mathbf{x}}_{\alpha\beta} = \mathbf{F}^{(0)}_{\alpha\beta}(\mathbf{x}_{\alpha\beta}, t). \tag{3.6}$$

The foregoing equation tells us flows on the tangential directions of the separation boundary surface. However, the boundary dynamical properties can be given or enforced in some discontinuous dynamical systems. Such given dynamical properties may not have the convexity (i.e., the nonconvex case), which can be referred to Aubin and Cellina (1984). For the given sliding dynamics on the boundary, the corresponding sliding flow is uniquely determined. From Theorems 2.1 and 2.3, the onset of sliding motions requires $\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-})\}\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m-})\} = 0$. However, whether the flow passes through from the domain Ω_α to Ω_β or not is also dependent on the dynamics of the discontinuous dynamical system in the neighborhood of the boundary. The onset of the sliding motion from the passable boundary is discussed first. Notice that the dynamics on the boundary surface can be given through a certain dynamic system.

3.2. Switching bifurcations

In this section, the switching bifurcation between the passable and nonpassable flows will be discussed. In addition, the switching bifurcation between the sink and source flows on the separation boundary will be discussed. The switching bifurcations are defined first, and then the sufficient and necessary conditions for such switching bifurcations will be developed. The product function of normal vector fields will be introduced to develop criteria for such switching bifurcations through the sufficient and necessary conditions.

DEFINITION 3.1. For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^r_{[t_{m+\varepsilon}, t_m)}$- and $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous ($r \geqslant 1$) for time t, respectively. The tangential bifurcation of the flow $\mathbf{x}^{(j)}(t)$ at $\mathbf{x}_m$ on the boundary $\overrightarrow{\partial\Omega}_{ij}$ is termed *the switching bifurcation of the first kind of nonpassable boundary* (or called *the sliding bifurcation*) if the following two conditions hold:

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(j)}(t_{m\pm}) = 0 \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(i)}(t_{m-}) \neq 0, \quad \text{and} \tag{3.7}$$

either

$$\begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(j)}(t_{m-\varepsilon})\right] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})\right] > 0 \end{cases} \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \tag{3.8}$$

or

$$\begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(j)}(t_{m-\varepsilon})\right] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})\right] < 0 \end{cases} \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \tag{3.9}$$

DEFINITION 3.2. For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$- and $C^r_{[t_{m+\varepsilon}, t_m)}$-continuous ($r \geqslant 1$) for time t, respectively. The tangential bifurcation of the flow $\mathbf{x}^{(i)}(t)$ with the flow $\mathbf{x}^{(j)}(t)$ at $\mathbf{x}_m$ on the boundary $\overrightarrow{\partial\Omega}_{ij}$ is termed *the switching bifurcation of the second kind of nonpassable boundary* (or called *the source bifurcation*) if the following two conditions hold:

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(i)}(t_{m\pm}) = 0 \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(j)}(t_{m+}) \neq 0, \quad \text{and} \tag{3.10}$$

either

$$\left.\begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})\right] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(i)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] > 0 \end{cases}\right. \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \tag{3.11}$$

or

$$\left.\begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})\right] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(i)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] < 0 \end{cases}\right. \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \tag{3.12}$$

From the above two definitions, the geometric illustrations for the switching bifurcation from the semi-passable to the nonpassable boundary of the first and second kinds are presented in Fig. 3.1. Before the sliding bifurcation occurs, the flow $\mathbf{x}^{(j)}(t)$ for $t \in [t_{m-\varepsilon}, t_{m-})$ cannot exist on the semi-passable boundary $\overrightarrow{\partial\Omega}_{ij}$. However, the flow $\mathbf{x}^{(i)}(t)$ for $t \in (t_{m+}, t_{m+\varepsilon}]$ cannot appear before the switching bifurcation. From the foregoing two definitions, the source or sink bifurcation on the semi-passable boundary requires the tangential bifurcation of the input or output flow. The switching bifurcation from of $\overrightarrow{\partial\Omega}_{ij}$ to $\overleftarrow{\partial\Omega}_{ij}$ can be defined as follows:

DEFINITION 3.3. For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\mp})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous ($r \geqslant 1$) for time t. The tangential bifurcation of the flow $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ at $\mathbf{x}_m$ on the boundary $\overrightarrow{\partial\Omega}_{ij}$ is termed *the switching bifurcation* of $\overrightarrow{\partial\Omega}_{ij}$ to $\overleftarrow{\partial\Omega}_{ij}$ if the following two conditions hold:

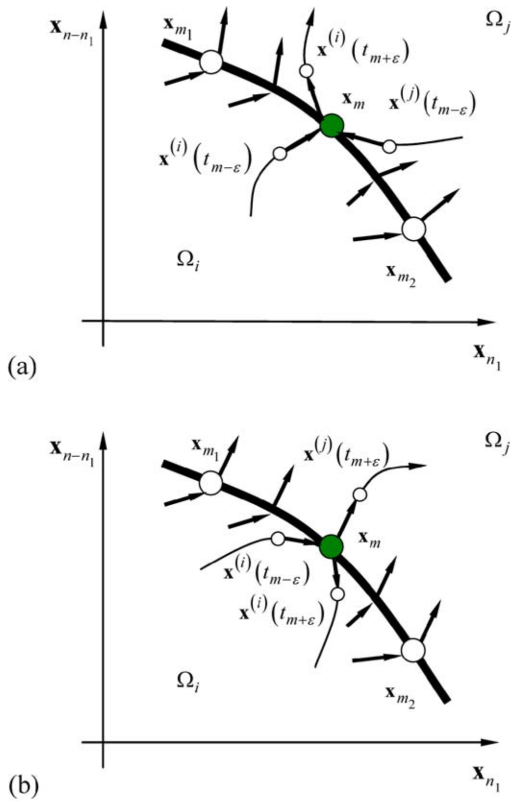


Figure 3.1. (a) The sliding bifurcation, and (b) the source bifurcation on the semi-passable boundary $\overrightarrow{\partial\Omega}_{ij}$. Four points $\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon})$ ($\alpha \in \{i, j\}$) and $\mathbf{x}_m$ lie in the corresponding domains and on the boundary $\partial\Omega_{ij}$, respectively.

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m\pm}) = 0 \quad \text{for } \alpha \in \{i, j\}, \quad \text{and} \tag{3.13}$$

either

$$\left\{\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})\right] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(i)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(j)}(t_{m-\varepsilon})\right] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] > 0 \end{array}\right. \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \tag{3.14}$$

or

$$\begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})\right] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(i)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(j)}(t_{m-\varepsilon})\right] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] < 0 \end{cases} \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \tag{3.15}$$

The above definitions give all the possible switching bifurcations of the semi-passable motion on $\overrightarrow{\partial\Omega}_{ij}$. The corresponding theorems can be stated as follows to determine the switching bifurcations. The proofs can be completed in a similar manner to Theorems 2.8–2.10. The necessary and sufficient conditions for the switching bifurcations will be provided.

THEOREM 3.1. *For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^r_{[t_{m+\varepsilon}, t_m)}$- and $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous ($r \geqslant 2$) for time t, respectively. The sliding bifurcation of the flow $\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ on the boundary $\overrightarrow{\partial\Omega}_{ij}$ exists iff*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m\pm}) = 0 \quad \textit{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-}) \neq 0; \tag{3.16}$$

$$\begin{aligned} &\textit{either} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-}) > 0 \quad \textit{for}\, \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j, \\ &\textit{or} \qquad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-}) < 0 \quad \textit{for}\, \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i; \end{aligned} \tag{3.17}$$

$$\begin{aligned} &\left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{F}^{(j)}(t_{m-\varepsilon}) - \mathbf{F}^{(0)}(t_{m-\varepsilon})\right]\right\} \\ &\quad \times \left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{F}^{(j)}(t_{m+\varepsilon}) - \mathbf{F}^{(0)}(t_{m-\varepsilon})\right]\right\} < 0, \quad \textit{or} \\ &\textit{either} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathrm{D}\mathbf{F}^{(j)}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}(t_{m\pm})\right] > 0 \quad \textit{for}\, \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ &\textit{or} \qquad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathrm{D}\mathbf{F}^{(j)}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}(t_{m\pm})\right] < 0 \quad \textit{for}\, \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{aligned} \tag{3.18}$$

PROOF. Following the proof procedures in Theorems 2.9–2.11, the above theorem can be easily proved. □

THEOREM 3.2. *For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$- and $C^r_{[t_{m+\varepsilon}, t_m)}$-continuous ($r \geqslant 2$) for time t, respectively. The source bifurcation of the flow*

$\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ *on the boundary* $\overrightarrow{\partial\Omega}_{ij}$ *exists iff*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m\pm}) = 0 \quad \textit{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+}) \neq 0; \tag{3.19}$$

$$\begin{aligned} &\textit{either} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+}) > 0 \quad \textit{for}\ \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ &\textit{or} \qquad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+}) < 0 \quad \textit{for}\ \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i; \end{aligned} \tag{3.20}$$

$$\begin{aligned} &\left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{F}^{(i)}(t_{m-\varepsilon}) - \mathbf{F}^{(0)}(t_{m-\varepsilon})\right]\right\} \\ &\quad \times \left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{F}^{(i)}(t_{m+\varepsilon}) - \mathbf{F}^{(i)}(t_{m+\varepsilon})\right]\right\} < 0, \quad \textit{or} \\ &\textit{either} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathrm{D}\mathbf{F}^{(i)}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}(t_{m\pm})\right] < 0 \quad \textit{for}\ \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ &\textit{or} \qquad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathrm{D}\mathbf{F}^{(i)}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}(t_{m\pm})\right] > 0 \quad \textit{for}\ \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{aligned} \tag{3.21}$$

PROOF. Following the proof procedures in Theorems 2.9–2.11, the above theorem can be easily proved. □

THEOREM 3.3. *For a discontinuous dynamical system in Eq.* (2.1)*, there is a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ *for* t_m*. For an arbitrarily small* $\varepsilon > 0$*, there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$*), and* $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\mp})$*. The flows* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *are* $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 2)$ *for time* t*. The switching bifurcation of the flow* $\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ *at* $\mathbf{x}_m$ *on the boundary* $\overrightarrow{\partial\Omega}_{ij}$ *exists iff*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m\pm}) = 0 \quad \textit{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m\pm}) = 0, \tag{3.22}$$

$$\begin{aligned} &\left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{F}^{(\alpha)}(t_{m-\varepsilon}) - \mathbf{F}^{(0)}(t_{m-\varepsilon})\right]\right\} \\ &\quad \times \left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{F}^{(\alpha)}(t_{m+\varepsilon}) - \mathrm{D}\mathbf{F}^{(0)}(t_{m+\varepsilon})\right]\right\} < 0 \quad \textit{for}\ \alpha \in \{i, j\}, \quad \textit{or} \end{aligned} \tag{3.23}$$

$$\begin{aligned} &\textit{either} \quad \begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathrm{D}\mathbf{F}^{(i)}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}(t_{m\pm})\right] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathrm{D}\mathbf{F}^{(j)}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}(t_{m\pm})\right] > 0 \end{cases} \quad \textit{for}\ \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ &\textit{or} \qquad \begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathrm{D}\mathbf{F}^{(i)}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}(t_{m\pm})\right] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathrm{D}\mathbf{F}^{(j)}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}(t_{m\pm})\right] < 0 \end{cases} \quad \textit{for}\ \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{aligned} \tag{3.24}$$

PROOF. Following the proof procedures in Theorems 2.9–2.11, the above theorem can be easily proved. □

DEFINITION 3.4. For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \partial\Omega_{ij}$ for t_m and $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$, $\alpha \in \{i, j\}$. *The product of the normal vector fields* on the boundary $\partial\Omega_{ij}$ is defined as

$$L_{\alpha\beta}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_\alpha, \boldsymbol{\mu}_\beta) = \left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(t_{m\mp})\right]\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}(t_{m\pm})\right], \tag{3.25}$$

where $\beta \in \{i, j\}$.

From the foregoing definitions, the passable, sink and source boundaries $\partial\Omega_{\alpha\beta}$, respectively, require the normal vector field products as

$$\begin{aligned} &L_{\alpha\beta}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_\alpha, \boldsymbol{\mu}_\beta) > 0 \quad \text{on } \overrightarrow{\partial\Omega}_{\alpha\beta}, \\ &L_{\alpha\beta}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_\alpha, \boldsymbol{\mu}_\beta) < 0 \quad \text{on } \overline{\partial\Omega}_{\alpha\beta} = \widetilde{\partial\Omega}_{\alpha\beta} \cup \widehat{\partial\Omega}_{\alpha\beta}. \end{aligned} \tag{3.26}$$

It is obviously observed that the switching bifurcation of the flow at $(\mathbf{x}_m, t_m)$ on the boundary $\partial\Omega_{\alpha\beta}$ requires the normal vector product function

$$L_{\alpha\beta}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_\alpha, \boldsymbol{\mu}_\beta) = 0. \tag{3.27}$$

If the product of the normal vector field is defined on one side of the neighborhood of the boundary $\partial\Omega_{\alpha\beta}$, we have

$$\begin{aligned} L_{\alpha\alpha}(\mathbf{x}_{m\pm\varepsilon}, t_{m\pm\varepsilon}, \boldsymbol{\mu}_\alpha) = {} & \left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{F}^{(\alpha)}(t_{m-\varepsilon}) - \mathbf{F}^{(0)}(t_{m-\varepsilon})\right]\right\} \\ & \times \left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{F}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{F}^{(0)}(t_{m+\varepsilon})\right]\right\}. \end{aligned} \tag{3.28}$$

If $L_{\alpha\alpha}(\mathbf{x}_{m\pm\varepsilon}, t_{m\pm\varepsilon}, \boldsymbol{\mu}_\alpha) < 0$ and $\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(t_{m-}) = 0$, the flow $\mathbf{x}^{(\alpha)}(t)$ at $(\mathbf{x}_m, t_m)$ is tangential to the boundary $\partial\Omega_{\alpha\beta}$.

Consider the normal vector field product varying with the parameters vector $\mathbf{p}_{ij} \in \{\boldsymbol{\mu}_\alpha\}_{\alpha\in\{i,j\}}$ for the switching motion of the semi-passable boundary $\overrightarrow{\partial\Omega}_{\alpha\beta}$ to the nonpassable boundary. The normal vector field products for the different location of the boundary are different. The normal vector field products between two points $\mathbf{x}_{m_1}$ and $\mathbf{x}_{m_2}$ on the boundary $\partial\Omega_{\alpha\beta}$ are depicted in Fig. 3.2 for the parameter vector $\mathbf{p}_{\alpha\beta}$ between $\mathbf{p}^{(1)}_{\alpha\beta}$ and $\mathbf{p}^{(2)}_{\alpha\beta}$. For a specific values $\mathbf{p}^{(cr)}_{\alpha\beta}$ between $\mathbf{p}^{(1)}_{\alpha\beta}$ and $\mathbf{p}^{(2)}_{\alpha\beta}$, there is a point $\mathbf{x}_m$ on the boundary for the switching bifurcation from the semi-passable to nonpassable motion on the separation boundary. Two points $\mathbf{x}_{k_1}$ and $\mathbf{x}_{k_2}$ are the onset and vanishing of the nonpassable boundary for parameter on the boundary $\partial\Omega_{\alpha\beta}$. The dashed and solid curves represent $L_{\alpha\beta} < 0$ and $L_{\alpha\beta} \geqslant 0$, respectively. For $\mathbf{p}_{\alpha\beta}$ varying from $\mathbf{p}^{(1)}_{\alpha\beta} \to \mathbf{p}^{(cr)}_{\alpha\beta}$, the normal vector field product at $\mathbf{x} \in (\mathbf{x}_{m_1}, \mathbf{x}_{m_2})$ on the boundary is positive (i.e., $L_{\alpha\beta} > 0$). Therefore, the boundary $\partial\Omega_{\alpha\beta}$ is semi-passable. For $\mathbf{p}_{\alpha\beta}$ varying from $\mathbf{p}^{(cr)}_{\alpha\beta} \to \mathbf{p}^{(2)}_{\alpha\beta}$, there are two values $L_{\alpha\beta} > 0$ for $\mathbf{x} \in \{[\mathbf{x}_{m_1}, \mathbf{x}_{k_1}), (\mathbf{x}_{k_2}, \mathbf{x}_{m_2}]\}$ and $L_{\alpha\beta} < 0$ for $\mathbf{x} \in (\mathbf{x}_{k_1}, \mathbf{x}_{k_2})$. From Eq. (3.25), the portion of $\mathbf{x} \in (\mathbf{x}_{k_1}, \mathbf{x}_{k_2})$ on the boundary $\partial\Omega_{\alpha\beta}$ is nonpassable. The portion of the boundary, relative to $L_{\alpha\beta} > 0$, is

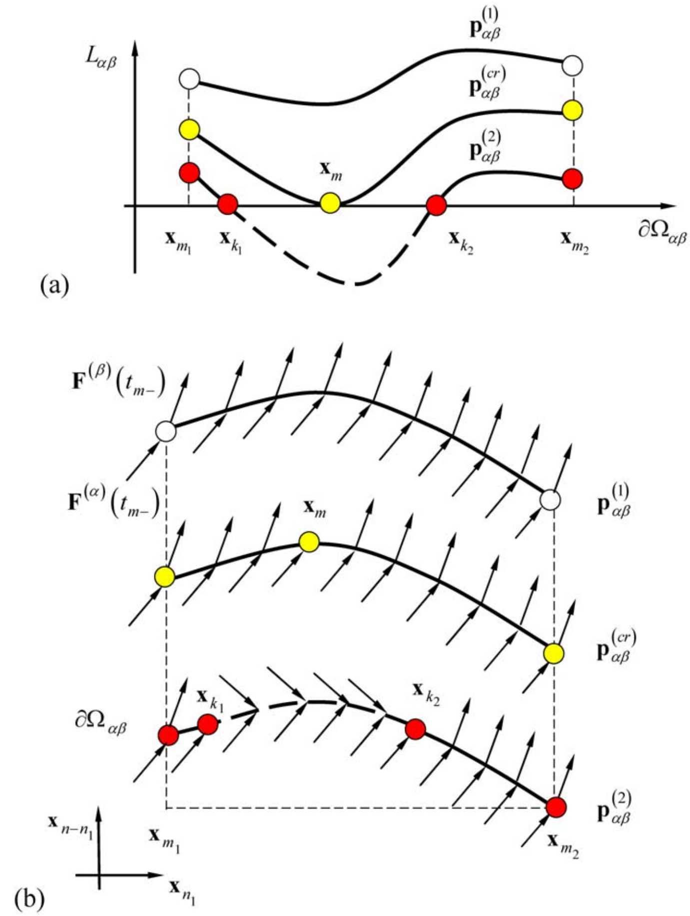


Figure 3.2. (a) The normal vector field product, and (b) the vector fields between two points $\mathbf{x}_{m1}$ and $\mathbf{x}_{m2}$ on the boundary $\partial\Omega_{\alpha\beta}$. The point $\mathbf{x}_m$ for $\mathbf{p}_{\alpha\beta}^{(cr)}$ is the critical point for the switching bifurcation. Two points $\mathbf{x}_{k1}$ and $\mathbf{x}_{k2}$ are the onset and vanishing of the nonpassable boundary for parameter on the boundary $\overrightarrow{\partial\Omega}_{\alpha\beta}$. The dashed and solid curves represent $L_{\alpha\beta} < 0$ and $L_{\alpha\beta} \geqslant 0$, respectively.

semi-passable. For parameter $\mathbf{p}_{\alpha\beta}$ varying from $\mathbf{p}_{\alpha\beta}^{(1)} \to \mathbf{p}_{\alpha\beta}^{(2)}$, the point $(\mathbf{x}_m, \mathbf{p}_{\alpha\beta}^{(cr)})$ on the boundary $\partial\Omega_{\alpha\beta}$ is the onset of the nonpassable boundary. However, for parameter $\mathbf{p}_{\alpha\beta}$ varying from $\mathbf{p}_{\alpha\beta}^{(2)} \to \mathbf{p}_{\alpha\beta}^{(1)}$, there is a point for the vanishing of the nonpassable boundary. For three critical points $\{\mathbf{x}_m, \mathbf{x}_{k_1}, \mathbf{x}_{k_2}\}$, the corresponding normal vector field product is zero (i.e., $L_{\alpha\beta} = 0$). For $L_{\alpha\beta}$ in Fig. 3.2(a), the corresponding vector fields varying with the system parameter on the boundary $\partial\Omega_{\alpha\beta}$ are illustrated in Fig. 3.2(b). $\mathbf{F}^{(\alpha)}(t_{m-})$ and $\mathbf{F}^{(\beta)}(t_{m\pm})$ are the limits of the vector fields in the domains Ω_α and Ω_β to the boundary $\partial\Omega_{\alpha\beta}$, respectively. This nonpassable boundary pertaining to $L_{\alpha\beta} < 0$ is the sink boundary. The critical points $\{\mathbf{x}_{k_1}, \mathbf{x}_{k_2}\}$ have the same properties as the point $\mathbf{x}_m$ for parameter $\mathbf{p}_{\alpha\beta}^{(cr)}$, (i.e., $L_{\alpha\beta}(\mathbf{x}_m) = 0$, $L_{\alpha\alpha}(\mathbf{x}_{m\pm\varepsilon}) < 0$ or $L_{\beta\beta}(\mathbf{x}_{m\pm\varepsilon}) > 0$). If the two critical points have the different properties, the sliding motion between the two different critical points will be discussed later. For instance, the normal vector field product functions are $L_{\alpha\beta}(\mathbf{x}_{k_1}) = 0$, $L_{\alpha\alpha}(\mathbf{x}_{k_1\pm\varepsilon}) < 0$ for the point $\mathbf{x}_k$ but $L_{\alpha\beta}(\mathbf{x}_{k_2}) = 0$, $L_{\beta\beta}(\mathbf{x}_{k_2\pm\varepsilon}) < 0$ for the point $\mathbf{x}_{k_2}$.

From the above theorems, the corresponding theorems are stated by using the normal vector field product function.

THEOREM 3.4. *For a discontinuous dynamical system in Eq.* (2.1)*, there is a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ *for* t_m*. For an arbitrarily small* $\varepsilon > 0$*, there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$*), and* $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$*. The flows* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *are* $C^r_{[t_{m+\varepsilon}, t_m)}$*- and* $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$*-continuous* ($r \geqslant 2$) *for time* t*, respectively. The sliding bifurcation of the flow* $\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ *on the boundary* $\overrightarrow{\partial\Omega}_{ij}$ *exists iff*

$$L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j) = 0, \tag{3.29}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-}) \neq 0 \quad \textit{and} \quad L_{jj}(\mathbf{x}_{m\pm\varepsilon}, t_{m\pm\varepsilon}, \boldsymbol{\mu}_j) < 0. \tag{3.30}$$

PROOF. Applying the normal vector field product in Definition 3.4 to Theorem 3.1, the foregoing theorem can be easily proved. □

THEOREM 3.5. *For a discontinuous dynamical system in Eq.* (2.1)*, there is a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ *for* t_m*. For an arbitrarily small* $\varepsilon > 0$*, there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$*), and* $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$*. The flows* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *are* $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$*- and* $C^r_{[t_{m+\varepsilon}, t_m)}$*-continuous* ($r \geqslant 2$) *for time* t*, respectively. The source bifurcation of the flow* $\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ *on the boundary* $\overrightarrow{\partial\Omega}_{ij}$ *exists iff*

$$L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j) = 0, \tag{3.31}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+}) \neq 0 \quad \textit{and} \quad L_{ii}(\mathbf{x}_{m\pm\varepsilon}, t_{m\pm\varepsilon}, \boldsymbol{\mu}_i) < 0. \tag{3.32}$$

PROOF. Applying the normal vector field product in Definition 3.4 to Theorem 3.2, the theorem can be easily proved. □

THEOREM 3.6. *For a discontinuous dynamical system in Eq.* (2.1)*, there is a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ *for* t_m*. For an arbitrarily small* $\varepsilon > 0$*, there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$*), and* $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\mp})$*. The flows* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *are* $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$*-continuous* ($r \geqslant 2$)

for time t. The switching bifurcation of the flow $\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ at $\mathbf{x}_m$ on the boundary $\overrightarrow{\partial\Omega}_{ij}$ exists iff

$$L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j) = 0, \tag{3.33}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m\pm}) = 0 \quad \textit{and} \quad L_{\alpha\alpha}(\mathbf{x}_{m\pm\varepsilon}, t_{m\pm\varepsilon}, \boldsymbol{\mu}_\alpha) < 0 \quad \textit{for } \alpha \in \{i, j\}. \tag{3.34}$$

PROOF. Applying the normal vector field product in Definition 3.4 to Theorem 3.3, the theorem can be easily proved. □

For the passable flow at $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ on the boundary $\overrightarrow{\partial\Omega}_{ij}$, consider the time interval $[t_{m_1}, t_{m_2}]$ for $[\mathbf{x}_{m_1}, \mathbf{x}_{m_2}]$ on the boundary, and the normal vector field product for $t_m \in [t_{m_1}, t_{m_2}]$ and $\mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}]$ is also positive, i.e., $L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j) > 0$. To determine the switching bifurcation, we define the local minimum of $L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j)$. Because $\mathbf{x}_m$ is a vector function of time t_m, the two total derivatives of $L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j)$ are introduced:

$$\begin{aligned} &\mathrm{D}L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j) \\ &\quad = \nabla L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j) \cdot \mathbf{F}^{(0)}_{ij}(\mathbf{x}_m, t_m) + \frac{\partial L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j)}{\partial t_m}, \end{aligned} \tag{3.35}$$

$$\mathrm{D}^k L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j) = \mathrm{D}^{k-1}\{\mathrm{D}L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j)\} \quad \text{for } k = 1, 2, \ldots. \tag{3.36}$$

Therefore, from Calculus, the local minimum of $L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j)$ is determined by

$$\mathrm{D}^k L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j) = 0 \quad (k = 1, 2, \ldots, 2l - 1), \tag{3.37}$$

$$\mathrm{D}^{2l} L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j) > 0. \tag{3.38}$$

DEFINITION 3.5. For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\mp})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous ($r \geqslant 1$) for time t. *The local minimum set* of the product of the normal vector fields (i.e., $L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j)$) is defined by

$$\begin{aligned} {}_{\min}L_{ij}(t_m) = \Big\{ & L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j) \mid \forall t_m \in [t_{m_1}, t_{m_2}],\ \exists \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}], \\ & \text{so that } \mathrm{D}^k L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j) = 0 \\ & \text{for } k = 1, 2, \ldots, 2l - 1, \text{ and } \mathrm{D}^{2l} L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j) > 0 \Big\}. \end{aligned} \tag{3.39}$$

From the local minimum set of $L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j)$, the global minimum of $L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j)$ can be defined as follows.

DEFINITION 3.6. For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\mp})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous ($r \geqslant 1$) for time t. The global minimum set of *the product of the normal vector fields* (i.e., $L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j)$) is defined by

$$_{\mathrm{Gmin}}L_{ij}(t_m) = \min_{t_m \in [t_{m_1}, t_{m_2}]} \big\{ {}_{\min}L_{ij}(t_m), L_{ij}(\mathbf{x}_{m_1}, t_{m_1}, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j), L_{ij}(\mathbf{x}_{m_2}, t_{m_2}, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j) \big\}. \tag{3.40}$$

From the foregoing definition, Theorems 3.4–3.6 can be expressed through the global minimum of $L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j)$. So we have the following corollaries. These corollaries will give the sufficient and necessary conditions for switching bifurcation onsets.

COROLLARY 3.1. *For a discontinuous dynamical system in Eq.* (2.1), *there is a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$), *and* $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$. *The flows* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *are* $C^r_{[t_{m+\varepsilon}, t_m)}$- *and* $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-*continuous* ($r \geqslant 2$) *for time t, respectively. The necessary and sufficient conditions for the sliding bifurcation of the flow* $\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ *on the boundary* $\overrightarrow{\partial\Omega}_{ij}$ *are*:

$$_{\mathrm{Gmin}}L_{ij}(t_m) = 0, \tag{3.41}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-}) \neq 0 \quad \textit{and} \quad L_{jj}(\mathbf{x}_{m\pm\varepsilon}, t_{m\pm\varepsilon}, \boldsymbol{\mu}_j) < 0. \tag{3.42}$$

COROLLARY 3.2. *For a discontinuous dynamical system in Eq.* (2.1), *there is a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$), *and* $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. *The flows* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *are* $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$- *and* $C^r_{[t_{m+\varepsilon}, t_m)}$-*continuous* ($r \geqslant 2$) *for time t, respectively. The necessary and sufficient conditions for the source bifurcation of the flow* $\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ *on the boundary* $\overrightarrow{\partial\Omega}_{ij}$ *are*:

$$_{\mathrm{Gmin}}L_{ij}(t_m) = 0, \tag{3.43}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+}) \neq 0 \quad \textit{and} \quad L_{ii}(\mathbf{x}_{m\pm\varepsilon}, t_{m\pm\varepsilon}, \boldsymbol{\mu}_i) < 0. \tag{3.44}$$

COROLLARY 3.3. *For a discontinuous dynamical system in Eq.* (2.1), *there is a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$), *and* $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\mp})$. *The flows* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *are* $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-*continuous* ($r \geqslant 2$) *for time t. The necessary and sufficient conditions for the switching bifurcation of* *the flow* $\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ *at* $\mathbf{x}_m$ *on the boundary* $\overrightarrow{\partial\Omega}_{ij}$ *are*:

$$_{\text{Gmin}}L_{ij}(t_m) = 0, \tag{3.45}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m\pm}) = 0 \quad \textit{and} \quad L_{\alpha\alpha}(\mathbf{x}_{m\pm\varepsilon}, t_{m\pm\varepsilon}, \boldsymbol{\mu}_\alpha) < 0 \quad \textit{for } \alpha \in \{i, j\}. \tag{3.46}$$

REMARK. The above corollaries are proved by replacing $L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j)$ through the global minimum values of $L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j)$ in the corresponding theorems.

3.3. Sliding fragmentation

The onset and vanishing of the sliding and source flows on the semi-passable boundary were discussed. The fragmentations of the sliding and source flows on the boundary are of great interest in this section. This kind of bifurcation is still a switching bifurcation. The definitions for such fragmentation bifurcations of flows on the nonpassable boundary are similar to the switching bifurcations from the semi-passable to nonpassable motion on the separation boundary. From a logic point of view, the necessary and sufficient conditions for the fragmentation bifurcation from the nonpassable to passable flow on the boundary are quite similar to the sliding and source bifurcations from the passable to nonpassable flow. However, the statements for the corresponding definitions and theorems have some modification. For clear description of the fragmentation bifurcation phenomena, the corresponding definitions and theorems are given as follows.

DEFINITION 3.7. For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widetilde{\partial\Omega}_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(\beta)}(t_{m\pm})$, $\alpha, \beta \in \{i, j\}$ and $\alpha \neq \beta$. The flows $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^r_{[t_{m+\varepsilon}, t_m)}$- and $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous ($r \geqslant 1$) for time t, respectively. The tangential bifurcation of the flow $\mathbf{x}^{(\beta)}(t)$ at $\mathbf{x}_m$ on the boundary $\widetilde{\partial\Omega}_{ij}$ is termed *the fragmentation bifurcation of the first kind* of nonpassable boundary (or called *the sliding fragmentation bifurcation*) if the following two conditions hold:

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m\pm}) = 0 \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) \neq 0, \quad \text{and} \tag{3.47}$$

either

$$\left\{\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m-\varepsilon})\right] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] > 0 \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] > 0 \end{array}\right. \tag{3.48}$$

or

$$\begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m-\varepsilon})\right] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] < 0 \end{cases} \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha. \tag{3.49}$$

DEFINITION 3.8. For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widehat{\partial\Omega}_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(\beta)}(t_{m+})$, $\alpha, \beta \in \{i, j\}$ and $\alpha \neq \beta$. The flows $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$- and $C^r_{[t_{m+\varepsilon}, t_m)}$-continuous ($r \geqslant 1$) for time t, respectively. The tangential bifurcation of the flow $\mathbf{x}^{(\alpha)}(t)$ with the flow $\mathbf{x}^{(\beta)}(t)$ at $\mathbf{x}_m$ on the boundary $\overrightarrow{\partial\Omega}_{ij}$ is termed *the fragmentation bifurcation of the second kind* of nonpassable boundary (or called *the source fragmentation bifurcation*) if the following two conditions hold:

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m\pm}) = 0 \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+}) \neq 0, \quad \text{and} \tag{3.50}$$

either

$$\begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] > 0 \end{cases} \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta \tag{3.51}$$

or

$$\begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] < 0 \end{cases} \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha. \tag{3.52}$$

For a geometrical explanation of the concept of the fragmentation bifurcation of the nonpassable motion on the separation boundary, the vector fields near the sink and source boundaries are sketched in Figs. 3.3 and 3.4, respectively. The switching from the sink or source boundary to the semi-passable boundary has two possibilities for each of them. Therefore, the corresponding conditions given in Definitions 3.5 and 3.6 have been changed accordingly. Before the fragmentation bifurcation of the nonpassable motion occurs on the separation boundary, the flows $\mathbf{x}^{(\alpha)}(t)$, $\alpha \in \{i, j\}$, exist for $t \in [t_{m-\varepsilon}, t_{m-})$ or $t \in (t_{m+}, t_{m+\varepsilon}]$ on the sink or source boundary. Only the sliding motion exists on such a boundary. After the fragmentation bifurcation occurs, the sliding motion on the separation boundary will split into at least two portions of the sliding and semi-passable motions. This phenomenon is called *the fragmentation of the sliding flow on the separation*

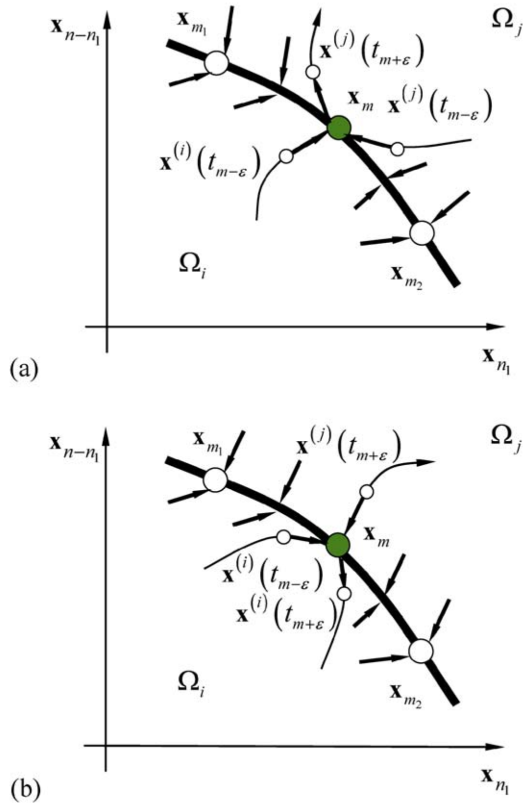


Figure 3.3. (a), (b) The sliding fragmentation bifurcation to the semi-passable boundary $\overrightarrow{\partial\Omega}_{\alpha\beta}$, $\alpha, \beta \in \{i, j\}$ and $\alpha \neq \beta$. Four points $\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon})$, $\mathbf{x}^{(\beta)}(t_{m\pm\varepsilon})$ and $\mathbf{x}_m$ lie in the corresponding domains and on the boundary $\partial\Omega_{ij}$, respectively.

boundary, which can help one easily understand the sliding dynamics on the separation boundary. In addition, for the nonpassable boundary, if the flows on both sides of the nonpassable boundary possess the local singularity at the boundary, the nonpassable boundary of the first kind switches into the nonpassable boundary of the second kind and vice versa. The local singularity of such a switching is similar to the one for the semi-passable motion on the separation boundary. The corresponding definition of the switching bifurcation is given as follows:

DEFINITION 3.9. For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widetilde{\partial\Omega}_{ij}$ (or $\widehat{\partial\Omega}_{ij}$) for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-

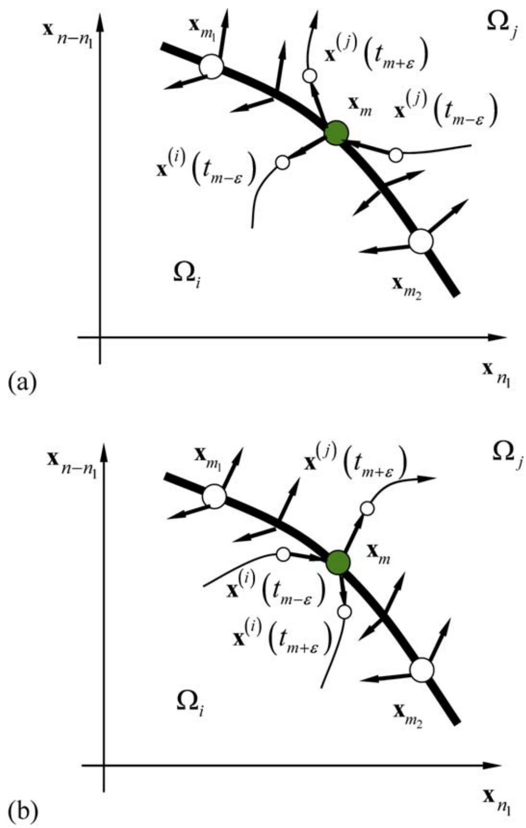


Figure 3.4. (a), (b) The source fragmentation bifurcation to the semi-passable boundary $\overrightarrow{\partial\Omega}_{\alpha\beta}$, $\alpha, \beta \in \{i, j\}$ and $\alpha \neq \beta$. Four points $\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon})$, $\mathbf{x}^{(\beta)}(t_{m\pm\varepsilon})$ and $\mathbf{x}_m$ lie in the corresponding domains and on the boundary $\partial\Omega_{ij}$, respectively.

continuous ($r \geqslant 1$) for time t. The tangential bifurcation of the flow $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ at $\mathbf{x}_m$ on the boundary $\widetilde{\partial\Omega}_{ij}$ (or $\widehat{\partial\Omega}_{ij}$) is termed the switching bifurcation of $\widetilde{\partial\Omega}_{ij}$ to $\widehat{\partial\Omega}_{ij}$ (or $\widehat{\partial\Omega}_{ij}$ to $\widetilde{\partial\Omega}_{ij}$) if the following two conditions hold:

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m\pm}) = 0 \quad \text{for } \alpha \in \{i, j\}, \quad \text{and} \tag{3.53}$$

either

$$\begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})\right] > 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(i)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] < 0; \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(j)}(t_{m-\varepsilon})\right] < 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] > 0 \end{cases} \tag{3.54}$$

for $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$ or

$$\begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})\right] < 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(i)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] > 0; \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(j)}(t_{m-\varepsilon})\right] > 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] < 0 \end{cases} \tag{3.55}$$

for $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i$.

The corresponding theorems for the fragmentation bifurcation of nonpassable motion on the boundary are given in a similar manner to Theorems 3.1–3.3.

THEOREM 3.7. *For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widetilde{\partial\Omega}_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(\beta)}(t_{m\pm})$, $\alpha, \beta \in \{i, j\}$ and $\alpha \neq \beta$. The flows $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^r_{[t_{m-\varepsilon}, t_m)}$- and $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous ($r \geqslant 1$) for time t, respectively. The sliding fragmentation bifurcation of the flow $\mathbf{x}^{(\alpha)}(t) \cup \mathbf{x}^{(\beta)}(t)$ on the boundary $\widetilde{\partial\Omega}_{ij}$ exists iff*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}(t_{m\pm}) = 0 \quad \textit{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-}) \neq 0; \tag{3.56}$$

$$\begin{aligned} &\textit{either} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-}) > 0 \quad \textit{for}\ \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta \\ &\textit{or} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-}) < 0 \quad \textit{for}\ \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha; \end{aligned} \tag{3.57}$$

$$\begin{aligned} &\left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{F}^{(\beta)}(t_{m-\varepsilon}) - \mathbf{F}^{(0)}(t_{m-\varepsilon})\right]\right\} \\ &\quad \times \left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{F}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{F}^{(0)}(t_{m+\varepsilon})\right]\right\} > 0, \quad \textit{or} \\ &\textit{either} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[D\mathbf{F}^{(\beta)}(t_{m\pm}) - D\mathbf{F}^{(0)}(t_{m\pm})\right] < 0 \quad \textit{for}\ \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta \\ &\textit{or} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[D\mathbf{F}^{(\beta)}(t_{m\pm}) - D\mathbf{F}^{(0)}(t_{m\pm})\right] < 0 \quad \textit{for}\ \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha. \end{aligned} \tag{3.58}$$

PROOF. Following the proof procedures in Theorems 2.9–2.11, the above theorem can be easily proved. □

THEOREM 3.8. *For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widehat{\partial\Omega}_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(\beta)}(t_{m+})$, $\alpha, \beta \in \{i, j\}$ and $\alpha \neq \beta$. The flows $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$- and $C^r_{(t_m, t_{m+\varepsilon}]}$-continuous ($r \geqslant 1$) for time t, respectively. The source* *fragmentation bifurcation of the flow* $\mathbf{x}^{(\alpha)}(t) \cup \mathbf{x}^{(\beta)}(t)$ *on the boundary* $\widehat{\partial\Omega}_{ij}$ *occurs iff*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m\pm}) = 0 \quad \textit{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}(t_{m+}) \neq 0; \tag{3.59}$$

$$\begin{aligned} &\textit{either} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}(t_{m+}) > 0 \quad \textit{for}\ \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta \\ &\textit{or} \qquad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}(t_{m+}) < 0 \quad \textit{for}\ \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha; \end{aligned} \tag{3.60}$$

$$\begin{aligned} &\big\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathbf{F}^{(\alpha)}(t_{m-\varepsilon}) - \mathbf{F}^{(0)}(t_{m-\varepsilon})\big]\big\} \\ &\quad \times \big\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathbf{F}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{F}^{(0)}(t_{m+\varepsilon})\big]\big\} < 0, \quad \textit{or} \\ &\textit{either} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[D\mathbf{F}^{(\alpha)}(t_{m\pm}) - D\mathbf{F}^{(0)}(t_{m\pm})\big] < 0 \quad \textit{for}\ \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta \\ &\textit{or} \qquad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[D\mathbf{F}^{(\alpha)}(t_{m\pm}) - D\mathbf{F}^{(0)}(t_{m\pm})\big] > 0 \quad \textit{for}\ \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha. \end{aligned} \tag{3.61}$$

PROOF. Following the proof procedures in Theorems 2.9–2.11, the above theorem can be easily proved. □

THEOREM 3.9. *For a discontinuous dynamical system in Eq. (2.1), there is a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widetilde{\partial\Omega}_{ij}$ *(or* $\widehat{\partial\Omega}_{ij}$*) for* t_m*. For an arbitrarily small* $\varepsilon > 0$*, there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$*), and* $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$*. The flows* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *are* $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 1)$ *for time* t*. The switching bifurcation of the flow from* $\widetilde{\partial\Omega}_{ij}$ *to* $\widehat{\partial\Omega}_{ij}$ *(or* $\widehat{\partial\Omega}_{ij}$ *to* $\widetilde{\partial\Omega}_{ij}$*) exists iff*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m\pm}) = 0 \quad \textit{for}\ \alpha \in \{i, j\}, \tag{3.62}$$

$$\begin{aligned} &\big\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathbf{F}^{(\alpha)}(t_{m-\varepsilon}) - \mathbf{F}^{(0)}(t_{m-\varepsilon})\big]\big\} \\ &\quad \times \big\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathbf{F}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{F}^{(0)}(t_{m+\varepsilon})\big]\big\} < 0 \quad \textit{for}\ \alpha \in \{i, j\}, \quad \textit{or} \end{aligned} \tag{3.63}$$

$$\textit{either} \quad \left\{\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[D\mathbf{F}^{(i)}(t_{m\pm}) - D\mathbf{F}^{(0)}(t_{m\pm})\big] < 0 \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[D\mathbf{F}^{(j)}(t_{m\pm}) - D\mathbf{F}^{(0)}(t_{m\pm})\big] > 0 \end{aligned}\right. \quad \textit{for}\ \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \tag{3.64a}$$

$$\textit{or} \quad \left\{\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[D\mathbf{F}^{(i)}(t_{m\pm}) - D\mathbf{F}^{(0)}(t_{m\pm})\big] > 0 \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[D\mathbf{F}^{(j)}(t_{m\pm}) - D\mathbf{F}^{(0)}(t_{m\pm})\big] < 0 \end{aligned}\right. \quad \textit{for}\ \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \tag{3.64b}$$

PROOF. Following the proof procedures in Theorems 2.9–2.11, the above theorem can be easily proved. □

In the similar manner, the normal vector field product varying with the parameters vector $\mathbf{p}_{ij} \in \{\boldsymbol{\mu}_\alpha\}_{\alpha\in\{i,j\}}$ is used to discuss the switching of the nonpassable boundary $\overline{\partial\Omega}_{\alpha\beta}$ to the semi-passable boundary $\overrightarrow{\partial\Omega}_{\alpha\beta}$. The normal vector products of the nonpassable boundary are $L_{\alpha\beta} < 0$ with varying with the location of the boundary. The normal vector field products between two points $\mathbf{x}_{m_1}$ and $\mathbf{x}_{m_2}$ on the sink boundary $\widetilde{\partial\Omega}_{\alpha\beta}$ are sketched in Fig. 3.5 for the parameter vector $\mathbf{p}_{\alpha\beta}$ between $\mathbf{p}_{\alpha\beta}^{(1)}$ and $\mathbf{p}_{\alpha\beta}^{(2)}$. For $L_{\alpha\beta}$ in Fig. 3.5(a), the corresponding vector fields varying with system parameter on the boundary $\partial\Omega_{\alpha\beta}$ are illustrated in Fig. 3.5(b). $\mathbf{F}^{(\alpha)}(t_{m-})$ and $\mathbf{F}^{(\beta)}(t_{m\pm})$ are the limits of the vector fields in the domains Ω_α and Ω_β to the boundary $\partial\Omega_{\alpha\beta}$, respectively. This nonpassable boundary pertaining to $L_{\alpha\beta} < 0$ is the sink boundary. There is a specific value $\mathbf{p}_{\alpha\beta}^{(cr)}$ between $\mathbf{p}_{\alpha\beta}^{(1)}$ and $\mathbf{p}_{\alpha\beta}^{(2)}$. Under this particular value, a point $\mathbf{x}_m$ on the sink boundary can be found for the sliding fragmentation bifurcation on the separation boundary. Two points $\mathbf{x}_{k_1}$ and $\mathbf{x}_{k_2}$ are the onset and vanishing of the passable boundary for parameter on the boundary $\widetilde{\partial\Omega}_{\alpha\beta}$. The dashed and solid curves represent $L_{\alpha\beta} > 0$ and $L_{\alpha\beta} \leqslant 0$, respectively. For $\mathbf{p}_{\alpha\beta}$ varying from $\mathbf{p}_{\alpha\beta}^{(1)} \to \mathbf{p}_{\alpha\beta}^{(cr)}$, the normal vector field product at $\mathbf{x} \in (\mathbf{x}_{m_1}, \mathbf{x}_{m_2})$ on the boundary is negative (i.e., $L_{\alpha\beta} < 0$). Therefore, the boundary $\partial\Omega_{\alpha\beta}$ is nonpassable. For $\mathbf{p}_{\alpha\beta}$ varying from $\mathbf{p}_{\alpha\beta}^{(cr)} \to \mathbf{p}_{\alpha\beta}^{(2)}$, there are two values $L_{\alpha\beta} < 0$ for $\mathbf{x} \in \{[\mathbf{x}_{m_1}, \mathbf{x}_{k_1}), (\mathbf{x}_{k_2}, \mathbf{x}_{m_2}]\}$ and $L_{\alpha\beta} > 0$ for $\mathbf{x} \in (\mathbf{x}_{k_1}, \mathbf{x}_{k_2})$. From Eq. (3.25), the portion of $\mathbf{x} \in (\mathbf{x}_{k_1}, \mathbf{x}_{k_2})$ on the boundary $\partial\Omega_{\alpha\beta}$ is semi-passable. The portion of the boundary, relative to $L_{\alpha\beta} > 0$, is semi-passable. For $\mathbf{p}_{\alpha\beta}$ varying from $\mathbf{p}_{\alpha\beta}^{(1)} \to \mathbf{p}_{\alpha\beta}^{(2)}$, the point $(\mathbf{x}_m, \mathbf{p}_{\alpha\beta}^{(cr)})$ on the boundary $\partial\Omega_{\alpha\beta}$ is the onset of the semi-passable boundary. It implies that the sliding motion on the separation boundary will be fragmentized. However, for parameter $\mathbf{p}_{\alpha\beta}$ varying from $\mathbf{p}_{\alpha\beta}^{(2)} \to \mathbf{p}_{\alpha\beta}^{(1)}$, the sliding fragmentation will disappear at some parameter value. For three critical points $(\mathbf{x}_m, \mathbf{x}_{k_1}, \mathbf{x}_{k_2})$, the corresponding normal vector field product is zero (i.e., $L_{\alpha\beta} = 0$). The critical points $\{\mathbf{x}_{k_1}, \mathbf{x}_{k_2}\}$ have the same properties as the critical point $\mathbf{x}_m$. If the two critical points have the different properties, the passable motion between the two different critical points will be discussed.

From the normal vector field product function, the criteria for the sliding fragmentation bifurcation can be given as similar to the ones in Theorems 3.4–3.6. Thus, the corresponding bifurcation conditions based on the normal vector field product function are stated as follows.

THEOREM 3.10. *For a discontinuous dynamical system in Eq.* (2.1), *there is a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widetilde{\partial\Omega}_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals* (*i.e.*, $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$), *and* $\mathbf{x}^{(\alpha)}(t_{m-}) =$

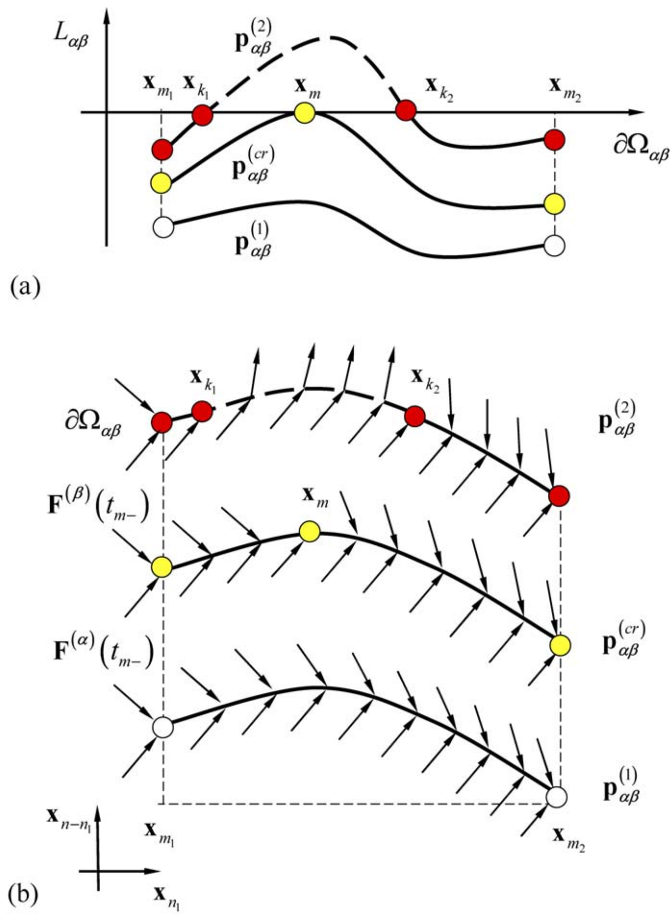


Figure 3.5. (a) The normal vector field product, and (b) the vector fields between two points $\mathbf{x}_{m_1}$ and $\mathbf{x}_{m_2}$ on the boundary $\widetilde{\partial\Omega}_{\alpha\beta}$. The point $\mathbf{x}_m$ for $\mathbf{p}_{\alpha\beta}^{(cr)}$ is the critical point for the switching bifurcation. Two points $\mathbf{x}_{k_1}$ and $\mathbf{x}_{k_2}$ are the onset and vanishing of the passable boundary for parameter on the boundary $\widetilde{\partial\Omega}_{\alpha\beta}$. The dashed and solid curves represent $L_{\alpha\beta} > 0$ and $L_{\alpha\beta} \leqslant 0$, respectively.

$\mathbf{x}_m = \mathbf{x}^{(\beta)}(t_{m\pm})$, *$\alpha, \beta \in \{i, j\}$ and $\alpha \neq \beta$. The flows $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^r_{[t_{m-\varepsilon}, t_m)}$- and $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous ($r \geqslant 1$) for time t, respectively. The sliding fragmentation bifurcation of the flow $\mathbf{x}^{(\alpha)}(t) \cup \mathbf{x}^{(\beta)}(t)$ on the boundary $\widetilde{\partial\Omega}_{ij}$ exists iff*

$$L_{\alpha\beta}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_\alpha, \boldsymbol{\mu}_\beta) = 0, \tag{3.65}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m+}) \neq 0 \quad \textit{and} \quad L_{\beta\beta}(\mathbf{x}_{m\pm\varepsilon}, t_{m\pm\varepsilon}, \boldsymbol{\mu}_j) < 0. \tag{3.66}$$

PROOF. Applying the normal vector field product in Definition 3.4 to Theorem 3.7, the foregoing theorem can be easily proved. □

THEOREM 3.11. *For a discontinuous dynamical system in Eq. (2.1), there is a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widehat{\partial\Omega}_{ij}$ *for* t_m*. For an arbitrarily small* $\varepsilon > 0$*, there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$*), and* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(\beta)}(t_{m+})$*,* $\alpha, \beta \in \{i, j\}$ *and* $\alpha \neq \beta$*. The flows* $\mathbf{x}^{(\alpha)}(t)$ *and* $\mathbf{x}^{(\beta)}(t)$ *are* $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$*- and* $C^r_{(t_m, t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 1)$ *for time t, respectively. The source fragmentation bifurcation of the flow* $\mathbf{x}^{(\alpha)}(t) \cup \mathbf{x}^{(\beta)}(t)$ *on the boundary* $\widehat{\partial\Omega}_{ij}$ *occurs iff*

$$L_{\alpha\beta}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_\alpha, \boldsymbol{\mu}_\beta) = 0, \tag{3.67}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}(t_{m+}) \neq 0 \quad \textit{and} \quad L_{\alpha\alpha}(\mathbf{x}_{m\pm\varepsilon}, t_{m\pm\varepsilon}, \boldsymbol{\mu}_j) < 0. \tag{3.68}$$

PROOF. Applying the normal vector field product in Definition 3.4 to Theorem 3.8, the foregoing theorem can be easily proved. □

THEOREM 3.12. *For a discontinuous dynamical system in Eq. (2.1), there is a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widetilde{\partial\Omega}_{ij}$ *(or* $\widehat{\partial\Omega}_{ij}$*) for* t_m*. For an arbitrarily small* $\varepsilon > 0$*, there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$*), and* $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$*. The flows* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *are* $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 1)$ *for time t. The switching bifurcation of the flow from* $\widetilde{\partial\Omega}_{ij}$ *to* $\widehat{\partial\Omega}_{ij}$ *(or* $\widehat{\partial\Omega}_{ij}$ *to* $\widetilde{\partial\Omega}_{ij}$*) exists iff*

$$L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j) = 0, \tag{3.69}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m\pm}) = 0 \quad \textit{and} \quad L_{\alpha\alpha}(\mathbf{x}_{m\pm\varepsilon}, t_{m\pm\varepsilon}, \boldsymbol{\mu}_\alpha) < 0 \quad \textit{for } \alpha \in \{i, j\}. \tag{3.70}$$

PROOF. Applying the normal vector field product in Definition 3.4 to Theorem 3.9, the foregoing theorem can be easily proved. □

For a nonpassable flow at $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widetilde{\partial\Omega}_{ij}$ (or $\widehat{\partial\Omega}_{ij}$) consider a time interval $[t_{m_1}, t_{m_2}]$ for $[\mathbf{x}_{m_1}, \mathbf{x}_{m_2}]$ on the boundary, and the normal vector field product for $t_m \in [t_{m_1}, t_{m_2}]$ and $\mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}]$ is also negative, i.e., $L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j) < 0$. To determine the switching bifurcation, we define the local maximum of $L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j)$. With Eqs. (3.35) and (3.36) from Calculus, the local maximum of $L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j)$ is determined by

$$\mathrm{D}^k L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j) = 0 \quad (k = 1, 2, \ldots, 2l - 1), \tag{3.71}$$

$$\mathrm{D}^{2l} L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j) < 0. \tag{3.72}$$

DEFINITION 3.10. For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widetilde{\partial\Omega}_{ij}$ (or $\widehat{\partial\Omega}_{ij}$) for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\mp})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous ($r \geqslant 1$) for time t. The local maximum set of the normal vector field product $L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j)$ is defined by

$$\begin{aligned} {}_{\max}L_{ij}(t_m) = \big\{ & L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j) \mid \forall t_m \in [t_{m_1}, t_{m_2}],\ \exists \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}], \\ & \text{so that } \mathrm{D}^k L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j) = 0 \\ & \text{for } k = 1, 2, \ldots, 2l-1, \text{ and } \mathrm{D}^{2l} L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j) > 0 \big\}. \end{aligned} \tag{3.73}$$

From the local maximum set of $L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j)$, the global maximum of $L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j)$ can be defined as follows.

DEFINITION 3.11. For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widetilde{\partial\Omega}_{ij}$ (or $\widehat{\partial\Omega}_{ij}$) for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\mp})$. The global maximum set of the normal vector field product $L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j)$ is defined by

$$\begin{aligned} {}_{\mathrm{Gmax}}L_{ij}(t_m) = \max_{t_m \in [t_{m_1}, t_{m_2}]} \big\{ & {}_{\max}L_{ij}(t_m), L_{ij}(\mathbf{x}_{m_1}, t_{m_1}, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j), \\ & L_{ij}(\mathbf{x}_{m_2}, t_{m_2}, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j) \big\}. \end{aligned} \tag{3.74}$$

From the foregoing definition, Theorems 3.10–3.12 can be expressed through the global maximum of $L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j)$. So we have the following corollaries. The corollaries will give the sufficient and necessary conditions for sliding fragmentation bifurcations.

COROLLARY 3.4. *For a discontinuous dynamical system in Eq.* (2.1)*, there is a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widetilde{\partial\Omega}_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$), *and* $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(\beta)}(t_{m\pm})$, $\alpha, \beta \in \{i, j\}$ *and* $\alpha \neq \beta$. *The flows* $\mathbf{x}^{(\alpha)}(t)$ *and* $\mathbf{x}^{(\beta)}(t)$ *are* $C^r_{[t_{m-\varepsilon}, t_m)}$*- and* $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$*-continuous* ($r \geqslant 1$) *for time t, respectively. The necessary and sufficient conditions for the sliding fragmentation bifurcation of the flow* $\mathbf{x}^{(\alpha)}(t) \cup \mathbf{x}^{(\beta)}(t)$ *on the boundary* $\widetilde{\partial\Omega}_{ij}$ *are*:

$${}_{\mathrm{Gmax}}L_{\alpha\beta}(t_m) = 0, \tag{3.75}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m+}) \neq 0 \quad \text{and} \quad L_{\beta\beta}(\mathbf{x}_{m\pm\varepsilon}, t_{m\pm\varepsilon}, \boldsymbol{\mu}_j) < 0. \tag{3.76}$$

COROLLARY 3.5. *For a discontinuous dynamical system in Eq.* (2.1)*, there is a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widehat{\partial\Omega}_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(\beta)}(t_{m+})$, $\alpha, \beta \in \{i, j\}$ and $\alpha \neq \beta$. The flows $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$- and $C^r_{(t_m, t_{m+\varepsilon}]}$-continuous ($r \geqslant 1$) for time t, respectively. The necessary and sufficient conditions for the source fragmentation bifurcation of the flow $\mathbf{x}^{(\alpha)}(t) \cup \mathbf{x}^{(\beta)}(t)$ on the boundary $\widehat{\partial\Omega}_{ij}$ are*:

$$_{\mathrm{Gmax}}L_{\alpha\beta}(t_m) = 0, \tag{3.77}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}(t_{m+}) \neq 0 \quad \text{and} \quad L_{\alpha\alpha}(\mathbf{x}_{m\pm\varepsilon}, t_{m\pm\varepsilon}, \boldsymbol{\mu}_j) < 0. \tag{3.78}$$

COROLLARY 3.6. *For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widetilde{\partial\Omega}_{ij}$ (or $\widehat{\partial\Omega}_{ij}$) for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous ($r \geqslant 1$) for time t. The necessary and sufficient conditions for the switching bifurcation of $\widetilde{\partial\Omega}_{ij}$ to $\widehat{\partial\Omega}_{ij}$ (or $\widehat{\partial\Omega}_{ij}$ to $\widetilde{\partial\Omega}_{ij}$) are*:

$$_{\mathrm{Gmax}}L_{ij}(t_m) = 0, \tag{3.79}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m\pm}) = 0 \quad \text{and} \quad L_{\alpha\alpha}(\mathbf{x}_{m\pm\varepsilon}, t_{m\pm\varepsilon}, \boldsymbol{\mu}_\alpha) < 0 \quad \text{for } \alpha \in \{i, j\}. \tag{3.80}$$

REMARK. The proof of the above corollaries can be completed by replacing $L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j)$ through its global maximum values in the corresponding theorems.

3.4. Sliding conditions in a friction oscillator

The friction-induced oscillator in Eqs. (2.62)–(2.66) is considered herein as an example to demonstrate how to develop the analytical conditions for sliding motion. The vectors in Eq. (2.67) are adopted. The corresponding subdomains and boundaries in Eq. (2.68) are used. The singular points are at $(\pm\infty, V)$. The equations of motion in Eqs. (2.64) and (2.66) can be described as

$$\dot{\mathbf{x}} = \mathbf{F}^{(\lambda)}(\mathbf{x}, t), \quad \lambda \in \{0, \alpha\} \tag{3.81}$$

where

$$\begin{aligned} &\mathbf{F}^{(\alpha)}(\mathbf{x}, t) = \big(y, F_\alpha(\mathbf{x}, \Omega t)\big)^{\mathrm{T}} \quad \text{in } \Omega_\alpha \ (\alpha \in \{1, 2\}); \\ &\mathbf{F}^{(0)}_{\alpha\beta}(\mathbf{x}, t) = (V, 0)^{\mathrm{T}} \quad \text{on } \widetilde{\partial\Omega}_{\alpha\beta} \ (\{\alpha, \beta\} \in \{1, 2\}, \alpha \neq \beta), \\ &\mathbf{F}^{(0)}_{\alpha\beta}(\mathbf{x}, t) \in \big[\mathbf{F}^{(\alpha)}(\mathbf{x}, t), \mathbf{F}^{(\beta)}(\mathbf{x}, t)\big] \quad \text{on } \overrightarrow{\partial\Omega}_{\alpha\beta}. \end{aligned} \tag{3.82}$$

Notice that $F_\alpha(\mathbf{x}, \Omega t)$ is given in Eq. (2.71). For a boundary $\widetilde{\partial\Omega}_{\alpha\beta}$ possessing nonzero measure, the critical initial and final states of the sliding motion are $(\Omega t_c, x_c, V)$ and $(\Omega t_f, x_f, V)$, respectively. Consider a sliding motion starting for $(\Omega t_i, x_i, V)$ and ending at $(\Omega t_{i+1}, x_{i+1}, V)$, where $(\Omega t_{i+1}, x_{i+1}, V) \triangleq (\Omega t_f, x_f, V)$. From Theorem 2.4, the sliding motion (or called the stick motion in physics) through the real flow is guaranteed for $t_m \in [t_i, t_{i+1}) \subseteq [t_c, t_f)$ by

$$\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(\mathbf{x}, t_{m-})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}(\mathbf{x}, t_{m-})\big] \leqslant 0. \tag{3.83}$$

For a boundary $\overrightarrow{\partial\Omega}_{\alpha\beta}$ with nonzero measure, the starting and ending states of the passable motion are $(\Omega t_s, x_s, V)$ and $(\Omega t_e, x_e, V)$ accordingly. From Theorem 2.2, the nonsliding motion (or called passable motion to the boundary in Luo (2005a, 2005d) is guaranteed for $t_n \subset (t_s, t_e)$ by

$$\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(\mathbf{x}, t_{n-})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}(\mathbf{x}, t_{n+})\big] > 0. \tag{3.84}$$

For the boundary switching from $\overrightarrow{\partial\Omega}_{\alpha\beta}$ to $\widetilde{\partial\Omega}_{\alpha\beta}$, we have $t_e = t_c$, otherwise, $t_f = t_s$. Suppose the starting state is a switching state of the sliding motion. From Theorem 2.4, the switching condition of the sliding motion from $\overrightarrow{\partial\Omega}_{\alpha\beta}$ to $\widetilde{\partial\Omega}_{\alpha\beta}$ at $t_m = t_c$ is

$$\begin{aligned}
&\text{either} \quad \big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(\mathbf{x}, t_{m-})\big] < 0, \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta,\\
&\text{or} \quad \big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(\mathbf{x}, t_{m-})\big] > 0, \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha;\\
&\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}(\mathbf{x}, t_{m+})\big] = 0.
\end{aligned} \tag{3.85}$$

Furthermore, for the sliding motion on $\widetilde{\partial\Omega}_{\alpha\beta}$, the sliding motion vanishing from $\widetilde{\partial\Omega}_{\alpha\beta}$ and going into the domain Ω_γ ($\gamma = \alpha$ or β) at $t_m = t_f$ requires:

$$\begin{aligned}
&\text{either} \quad \big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(\mathbf{x}, t_{m-})\big] < 0, \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta,\\
&\text{or} \quad \big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(\mathbf{x}, t_{m-})\big] > 0, \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha;\\
&\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\gamma)}(\mathbf{x}, t_{m+})\big] = 0.
\end{aligned} \tag{3.86}$$

From Eqs. (3.85) and (3.86), the switching conditions for the sliding motions are summarized as

$$\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(\mathbf{x}, t_{m-})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}(\mathbf{x}, t_{m-})\big] = 0. \tag{3.87}$$

The normal vector of the boundary $\partial\Omega_{12}$ and $\partial\Omega_{21}$ is given in Eq. (2.74). Therefore, we have

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(\mathbf{x}, t) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\beta\alpha}} \cdot \mathbf{F}^{(\alpha)}(\mathbf{x}, t) = F_\alpha(\mathbf{x}, \Omega t). \tag{3.88}$$

The normal projection of the vector flow is the force for the oscillator in Eq. (3.81). For a better understanding of the force characteristic of the sliding motion, the conditions for sliding and nonsliding motions in Eqs. (3.83) and (3.84) can be re-written by

$$F_1(\mathbf{x}_m, \Omega t_{m-}) < 0 \quad \text{and} \quad F_2(\mathbf{x}_m, \Omega t_{m-}) > 0 \quad \text{for } \partial\widetilde{\Omega}_{12}; \tag{3.89}$$

and

$$\begin{aligned} &F_1(\mathbf{x}_m, \Omega t_{m-}) < 0 \quad \text{and} \quad F_2(\mathbf{x}_m, \Omega t_{m+}) < 0 \quad \text{for } \partial\overrightarrow{\Omega}_{12}, \\ &F_2(\mathbf{x}_m, \Omega t_{m-}) > 0 \quad \text{and} \quad F_1(\mathbf{x}_m, \Omega t_{m+}) > 0 \quad \text{for } \partial\overrightarrow{\Omega}_{21}. \end{aligned} \tag{3.90}$$

Equations (3.85) and (3.86), respectively, give the switching conditions for the sliding motion in Eq. (2.83), i.e.,

$$\begin{aligned} &F_1(\mathbf{x}_m, \Omega t_{m-}) < 0 \quad \text{and} \quad F_2(\mathbf{x}_m, \Omega t_{m-}) = 0 \quad \text{for } \partial\overrightarrow{\Omega}_{12} \to \partial\widetilde{\Omega}_{12}, \\ &F_2(\mathbf{x}_m, \Omega t_{m-}) > 0 \quad \text{and} \quad F_1(\mathbf{x}_m, \Omega t_{m-}) = 0 \quad \text{for } \partial\overrightarrow{\Omega}_{21} \to \partial\widetilde{\Omega}_{21} \end{aligned} \tag{3.91}$$

at $t_m = t_c$, and

$$\begin{aligned} &F_1(\mathbf{x}_m, \Omega t_{m-}) < 0 \quad \text{and} \quad F_2(\mathbf{x}_m, \Omega t_{m-}) = 0 \quad \text{for } \partial\widetilde{\Omega}_{\alpha\beta} \to \Omega_2, \\ &F_2(\mathbf{x}_m, \Omega t_{m-}) > 0 \quad \text{and} \quad F_1(\mathbf{x}_m, \Omega t_{m-}) = 0 \quad \text{for } \partial\widetilde{\Omega}_{\alpha\beta} \to \Omega_1 \end{aligned} \tag{3.92}$$

at $t_m = t_f$. A sketch of the vector field and classification of the sliding motion is illustrated in Fig. 3.6. In Fig. 3.6(a), the switching condition for the sliding motion is presented through the vector fields of $\mathbf{F}^{(1)}(\mathbf{x}, t)$ and $\mathbf{F}^{(2)}(\mathbf{x}, t)$. The ending condition for sliding (or stick) motion along the velocity boundary is illustrated by $F_2(t_{m-}) = 0$. However, the starting point of the sliding motion may not be switching points from the possible boundary to the sliding motion boundary. When the flow arrives to the discontinuous boundary, once Eq. (3.89) holds, the sliding motion along the corresponding discontinuous boundary will be formed. From the switching conditions in Eqs. (3.91) and (3.92), there are four possible sliding motions (I)–(IV), as shown in Fig. 3.6(b). The switching condition in Eq. (3.91) is the critical condition for formation of the sliding motion. The above switching conditions for onset, forming and vanishing of the sliding motions along the discontinuous friction boundary at a certain velocity are strongly dependent on the total force acting on the oscillator. Because the total force can be contributed from the linear or nonlinear continuous forces from spring or dampers, it implies that such switching conditions can be applied to dynamical systems possessing nonlinear, continuous spring and viscous damping forces with a nonlinear friction with a C^0-discontinuity.

3.5. Sliding criteria for a friction oscillator

The three generic mappings in Eq. (2.83) are governed by the algebraic equations in Eq. (2.85). For nonsliding mapping, two equations are given. Two equations

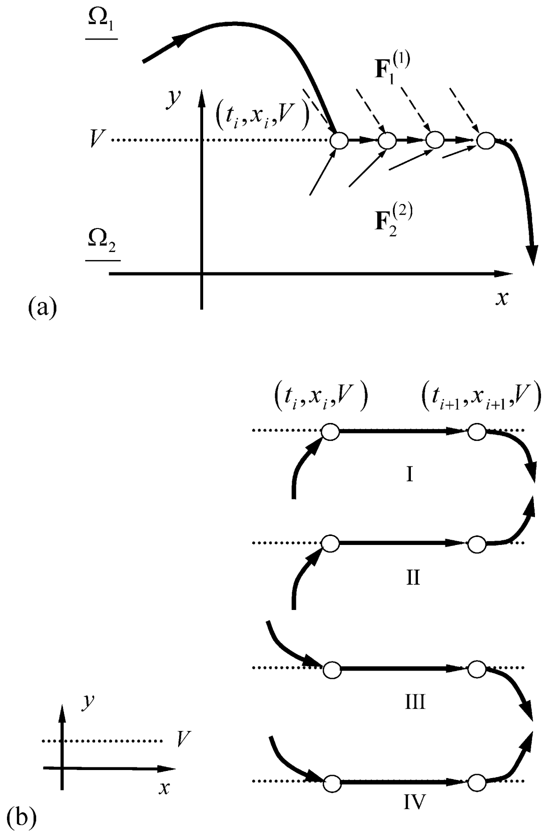


Figure 3.6. (a) The vector field, and (b) classification of sliding motions for belt speed $V > 0$.

with four unknowns in Eq. (2.85) cannot give the unique solutions for the sliding motion. Once the initial state is given, the final state of the sliding motion is uniquely determined. Consider the switching states $(\Omega t_c, x_c, V)$ and $(\Omega t_f, x_f, V)$ as the critical-initial and final conditions of the sliding motion, respectively. For the time interval $[t_i, t_{i+1}] \subseteq [t_c, t_f]$ of any sliding motion, the sliding motion requires $(\Omega t_{i+1}, x_{i+1}, V) \triangleq (\Omega t_f, x_f, V)$. From Eqs. (3.89) and (3.91), the initial condition of the sliding motion for all $t_i \in [t_c, t_f)$ and on the boundary $\partial\widetilde{\Omega}_{\alpha\beta}$ satisfies the following force relation:

$$L_{12}(t_i) = F_1(x_i, V, \Omega t_i) F_2(x_i, V, \Omega t_i) \leqslant 0. \tag{3.93}$$

The foregoing equation is the normal vector product of Eq. (3.81). The switching condition in Eq. (3.86) or (3.91) for the sliding motion at the critical time t_i also

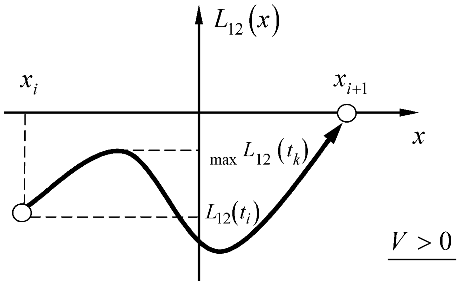


Figure 3.7. A sketch of a force product for sliding motion under given parameters for belt speed $V > 0$. $_{\max}L_{12}(t_k)$ and $L_{12}(t_i)$ represent the local *maximum* and *initial* force products, respectively. x_i and x_{i+1} represent the switching displacements of the stating and vanishing points of the sliding motion, respectively.

gives the *initial force product condition*, i.e.,

$$L_{12}(t_c) = F_1(x_c, V, \Omega t_c) F_2(x_c, V, \Omega t_c) = 0. \tag{3.94}$$

If the initial condition satisfies Eq. (3.94), the sliding motion is called the critical sliding motion. The condition in Eq. (3.86) or (3.92) for the sliding motion at the time $t_{i+1} = t_f$ gives the *final force product condition*, i.e.,

$$L_{12}(t_{i+1}) = F_1(x_{i+1}, V, \Omega t_{i+1}) F_2(x_{i+1}, V, \Omega t_{i+1}) = 0. \tag{3.95}$$

From Eq. (3.95), it is observed that the force product on the discontinuous boundary is also very significant. Once the force product of the sliding motion for $t_m \in (t_i, t_{i+1})$ changes its sign, the sliding motion for the friction induced oscillator will vanish. So the characteristic of the force product for the sliding motion should be further discussed. To explain the force mechanism of the sliding motion, the force product for sliding motion starting at $(\Omega t_i, x_i, V)$ and ending at $(\Omega t_{i+1}, x_{i+1}, V)$ is sketched in Fig. 3.7 for given parameters with the belt speed $V > 0$. x_i and x_{i+1} represent the switching displacements of the stating and ending points of the sliding motion, respectively.

From the local normal vector product, the local peak force product relative to the domains Ω_1 and Ω_2, is defined as

$$\begin{aligned} {}_{\max}L_{12}(t_k) = \big\{ & L_{12}(t_k) \mid \forall t_k \in (t_i, t_{i+1}),\ \exists \mathbf{x}_k \in (\mathbf{x}_i, \mathbf{x}_{i+1}), \\ & \mathrm{D}L_{12}(t)|_{t=t_k} = 0 \text{ and } \mathrm{D}^2 L_{12}(t)|_{t=t_k} < 0 \big\}, \end{aligned} \tag{3.96}$$

and from the global normal vector field product, the maximum force product for the computational convenience is defined as

$$_{\mathrm{Gmax}}L_{12}(t_k) = \max_{t_k \in (t_i, t_{i+1})} \big\{ L_{12}(t_i), {}_{\max}L_{12}(t_k) \big\}. \tag{3.97}$$

By using the chain rule and $\mathrm{d}x/\mathrm{d}t = V$ for the sliding motion, the above definitions are described as

$$
\begin{aligned}
{}_{\max}L_{12}(x_k) = \Big\{ L_{12}(x_k) \mid \forall x_k \in (x_i, x_{i+1}),\ \exists t_k = \tfrac{x_k - x_i}{V} \in [t_i, t_{i+1}], \\
\tfrac{\mathrm{d}}{\mathrm{d}x} L_{12}(x)|_{x=x_k} = 0 \text{ and } \tfrac{\mathrm{d}^2}{\mathrm{d}x^2} L_{12}(x)|_{x=x_k} < 0 \Big\},
\end{aligned} \tag{3.98}
$$

$$
{}_{\mathrm{Gmax}}L_{12}(x_k) = \max_{x_k \in (x_i, x_{i+1})} \big\{ L_{12}(x_i),\ {}_{\max}L_{12}(x_k) \big\}. \tag{3.99}
$$

Once one of the global maximum force products is greater than zero, the sliding motion will disappear. Further, the sliding motion will be fragmentized. The corresponding critical condition is

$$
{}_{\mathrm{Gmax}}L_{12}(t_k) = 0 \quad \text{or} \quad {}_{\mathrm{Gmax}}L_{12}(x_k) = 0, \quad \text{for } t_k \in (t_i, t_{i+1}). \tag{3.100}
$$

The foregoing condition is termed the *global maximum force product condition* of the sliding fragmentation. For simplicity, it is also called the *sliding fragmentation condition*. After fragmentation, consider the starting to ending points of the two sliding motions to be $(\Omega t_i, x_i, V)$ to $(\Omega t_{i+1}, x_{i+1}, V)$ and $(\Omega t_{i+2}, x_{i+2}, V)$ to $(\Omega t_{i+3}, x_{i+3}, V)$, respectively. It is assumed that $t_i < t_{i+1} \leqslant t_{i+2} < t_{i+3}$. The nonsliding motion between the two fragmentized motions requires from Eq. (3.84)

$$
L_{12}(t_k) > 0 \quad \text{for } t_k \in (t_{i+1}, t_{i+2}). \tag{3.101}
$$

The inverse process of the sliding motion fragmentation is the merging of the two adjacent sliding motions. If the two sliding motions merge together, the global maximum force product condition in Eq. (3.100) will be satisfied at the gluing point of the two sliding motions. Once the force products of the two sliding motions are monitored, the peak force product condition similar to Eq. (3.100) can be observed. Before the merging of the two sliding motions, we consider them as in fragmentized sliding motions. That is, $(\Omega t_{i+1}, x_{i+1}, V) \neq (\Omega t_{i+2}, x_{i+2}, V)$. Extending definitions of Eqs. (3.97) and (3.99) to the starting and ending points of the sliding motions. The merging condition for the two adjacent sliding motions is

$$
\begin{aligned}
&{}_{\mathrm{Gmax}}L_{12}(t_k) = 0 \quad \text{or} \quad {}_{\mathrm{Gmax}}L_{12}(x_k) = 0 \quad \text{for } k \in \{i+1, i+2\}, \\
&(\Omega t_{i+1}, x_{i+1}, V) = (\Omega t_{i+2}, x_{i+2}, V).
\end{aligned} \tag{3.102}
$$

To explain the above condition of the fragmentation and merging of the sliding motions, the corresponding force product characteristics and phase plane of the sliding motion are sketched in Fig. 3.8. The sliding, critical sliding and sliding fragmentation motions are depicted for given parameters with belt speed $V > 0$

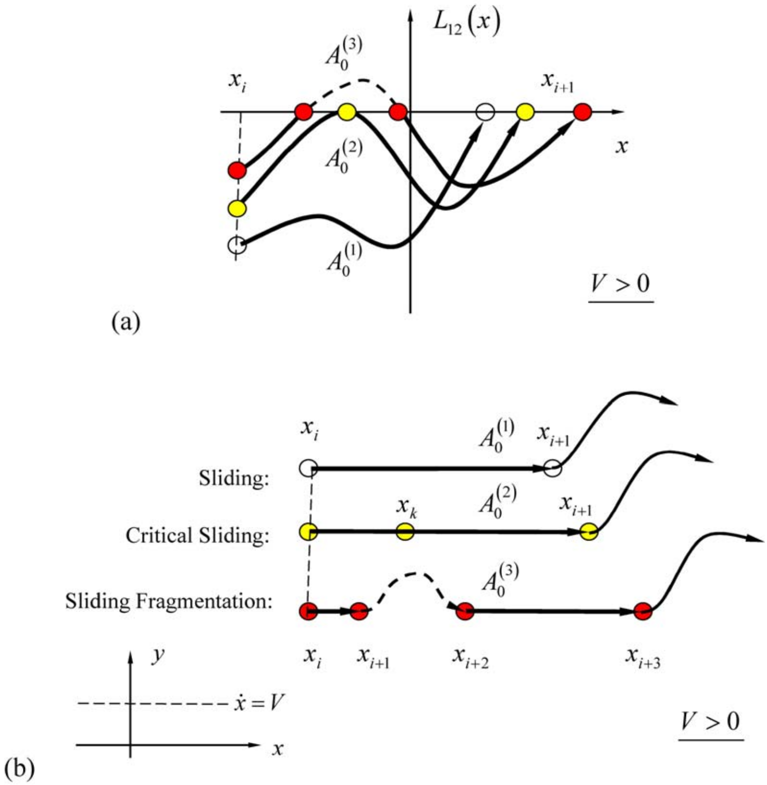


Figure 3.8. A sketch of (a) force product and (b) phase plane for sliding, critical sliding and sliding fragmentation motions under given parameters for belt speed $V > 0$ and $A_0^{(1)} < A_0^{(2)} < A_0^{(3)}$. The ending point of the sliding motion vanishes on the boundary and goes to the domain Ω_1.

and excitation amplitude $A_0^{(1)} < A_0^{(2)} < A_0^{(3)}$. The sliding motion vanishes from the boundary and goes into the domain Ω_1. The displacements x_i and x_{i+1} represent the starting and vanishing switching ones of the sliding motion, respectively. The point $x_k \in (x_i, x_{i+1})$ represents the critical point for the fragmentation or merging of the sliding motion. After the fragmentation (or before the merging), the critical point x_k is split into two new points x_{i+1} and x_{i+2}. However, after fragmentation (or merging), the index of x_{i+1} (or x_{x+3}) will be shifted as x_{x+3} (or x_{i+1}). Suppose the peak force product increases with increasing excitation amplitude. The motions from the sliding to the sliding fragmentation are given for $A_0^{(1)} \overset{\text{increase}}{\longrightarrow} A_0^{(3)}$. The dashed and thin solid curves represent nonsliding motion. It is clear that the nonsliding portion of the sliding fragmentation needs $L_{12}(t) > 0$ as in Eq. (3.103) for $t \in U_n = (t_j^n, t_{j+1}^n) \subset (t_i, t_{i+1})$ where $L_{12}(t_k) = 0$ for

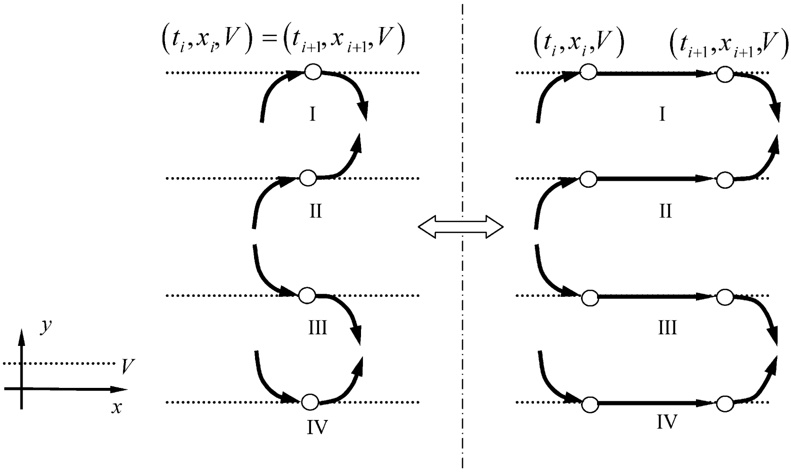


Figure 3.9. The four onsets of the sliding motion and the corresponding sliding motion in phase plane for belt speed $V > 0$.

$t_k = \{t_j^n, t_{j+1}^n\}$. However, for the sliding portion, the force products keep the relation $L_{12}(t) < 0$ for $t \in (t_i, t_{i+1}) \setminus \bigcup_n U_n$. Otherwise, suppose the peak force products at the potential merging points of the two sliding motion decrease with decreasing excitation. The merging of the two sliding motion can be observed for $A_0^{(3)} \overset{\text{decreases}}{\longrightarrow} A_0^{(1)}$. Whatever the merging or fragmentation of the sliding motion occur, the old sliding motion will be destroyed. Therefore, the conditions for the fragmentation and merging of the sliding motion are called the *vanishing* conditions for the sliding motion.

Consider the sliding motion with the nonzero measure of $\widetilde{\partial\Omega}_{\alpha\beta}$ with starting and ending points $(\Omega t_i, x_i, V)$ and $(\Omega t_{i+1}, x_{i+1}, V)$. Define $\delta = \sqrt{\delta_{x_i}^2 + \delta_{t_i}^2}$ with $\delta_{x_i} = x_{i+1} - x_i$ and $\delta_{t_i} = \Omega t_{i+1} - \Omega t_i \geqslant 0$. Suppose the starting and ending points satisfy the initial and final force product conditions, as $\delta \to 0$, the following force product condition is termed the *onset* condition for the sliding motion:

$$
\begin{aligned}
&L_{12}(t_k) = 0 \quad \text{for } k \in \{i, i+1\}, \\
&(\Omega t_{i+1}, x_{i+1}, V) = \lim_{\delta \to 0} (\Omega t_i + \delta_{t_i}, x_i + \delta_{x_i}, V).
\end{aligned}
\tag{3.103}
$$

The foregoing condition is also called *the onset force product condition*.

The onset condition has four possible cases from Eqs. (3.91) and (3.92), as sketched in Fig. 3.9. The four cases are: $F_2(t_i) = F_2(t_{i+1}) = 0$ (case I), $F_2(t_i) = F_1(t_{i+1}) = 0$ (case II), $F_1(t_i) = F_2(t_{i+1}) = 0$ (case III) and $F_1(t_i) = F_1(t_{i+1}) = 0$ (case IV). For $\delta \neq 0$, the four sliding motions exist. The onset conditions of the sliding motion for cases I and IV is the same as for the grazing motion. The details can be referred to Luo and Gegg (2006d). The two onsets of the sliding motions can be called the grazing onsets. The other onset conditions based on the cases II and III are the *inflexed* onsets for the sliding motions. In this subsection, the force product criteria for onset, forming and vanishing of the sliding motions along the discontinuous friction boundary at a certain velocity have been developed. From the expressions of such criteria, the achieved criteria can be applied for a nonlinear, continuous spring and viscous damper oscillator including a nonlinear friction with a C^0-discontinuity.

For the sliding motion vanishing on the boundary and going into the domain Ω_1, due to the trigonometric constraint $|\cos\Omega t_{i+1}| \leqslant 1$, the second equation of Eq. (2.84) gives

$$\begin{aligned} &|2d_1 V + c_1 x_{i+1} + b_1| \leqslant A_0 \quad \text{or} \\ &\left|2d_1 V + c_1\left[V(t_{i+1} - t_i) + x_i\right] + b_1\right| \leqslant A_0. \end{aligned} \tag{3.104}$$

Therefore, the final and initial switching displacements should be bounded by the foregoing equation. The criteria for the sliding motion are illustrated first through effects of excitation amplitude. Consider a set of system parameters $\{V = 1, \Omega = 1, d_1 = 1, d_2 = 0, b_1 = -b_2 = 30, c_1 = c_2 = 30, \mathrm{mod}(\Omega t_i, 2\pi) = \pi\}$. The final state responses, initial and maximum force products for specified initial displacements (i.e., $x_i = -2, -3, \ldots, -7$) are illustrated by the dark solid curves in Fig. 3.10. The onset and vanishing boundaries of the sliding motion are presented by the thin dashed curves. The boundaries of the sliding motion in the parameter space may be determined by the force product conditions. With varying the excitation amplitude, the final switching displacement and phases are presented in Figs. 3.10(a) and (b) for the sliding motion. The lower boundary is determined by the initial force product condition in Eq. (3.94), and the upper right-hand-side boundary is controlled by the sliding fragmentation condition in Eq. (3.100). The upper left-hand-side boundary is controlled by the merging force product conditions in Eq. (3.102). In Fig. 3.10(a), the dark, dashed lines are given by Eq. (3.104). The final switching displacement should lies in the right-hand-side area of the two dark dashed lines. The narrow strip for sliding motion varying with excitation amplitude is observed. With increasing excitation amplitude, the bandwidths of the final displacements and phases for the sliding motion do not change too much. The initial and maximum force products versus the excitation amplitude are shown in Figs. 3.10(c) and (d). The gray and hollow symbols represent the zero values for the initial and maximum force products, respectively. The force product plots (or the normal vector product) give the implications of the onset and vanishing of the sliding motion. For the shaded area, such a sliding motion does not exist any more. The analytical prediction from the theory gives

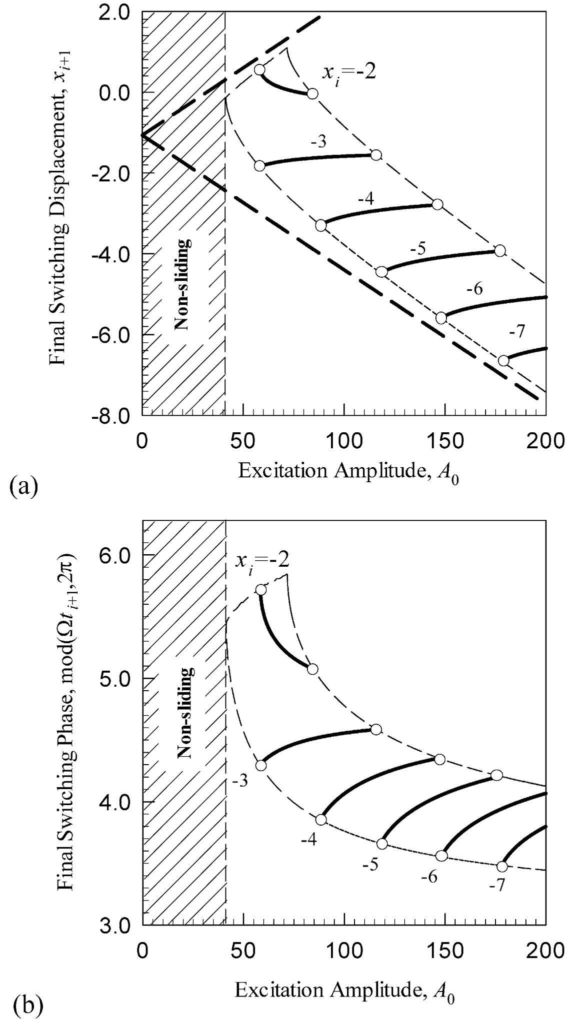
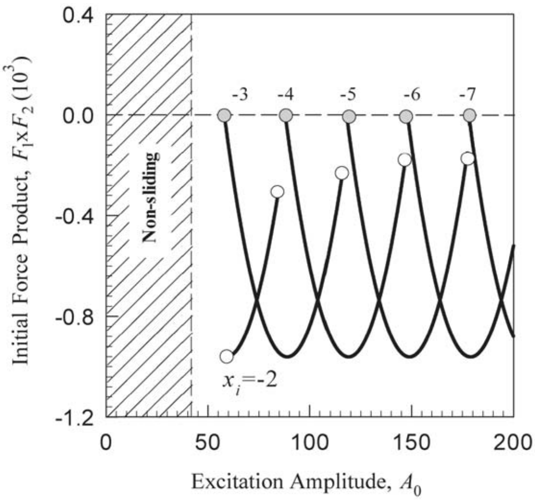
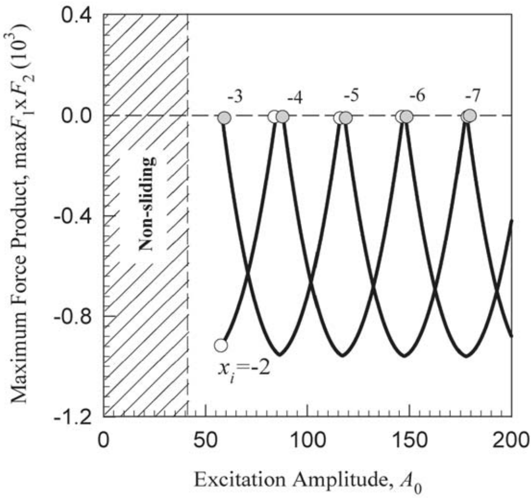


Figure 3.10. Effects of excitation amplitude for sliding motion vanishing on the boundary and going into the domain Ω_1: (a) final switching displacement, (b) final switching phase, (c) initial force product, and (d) maximum force product. ($V = 1, \Omega = 1, d_1 = 1, d_2 = 0, b_1 = -b_2 = 30, c_1 = c_2 = 30$, $\mathrm{mod}(\Omega t_i, 2\pi) = \pi$.)

the region for the sliding motion on the discontinuous boundary. Once the initial conditions of the motion get into such a domain, the sliding motion will be observed. Numerical illustrations of sliding motions will be presented.



(c)



(d)

Figure 3.10. (*continued*)

Due to discontinuity, the system possessing different parameters in the two domains will cause the different sliding characteristics. The constraint in Eq. (3.104) for the sliding motion going to the domain Ω_2 becomes

$$|2d_2V + c_2x_{i+1} + b_2| \leqslant A_0 \quad \text{or}$$
$$\left|2d_2V + c_2\left[V(t_{i+1} - t_i) + x_i\right] + b_2\right| \leqslant A_0. \tag{3.105}$$

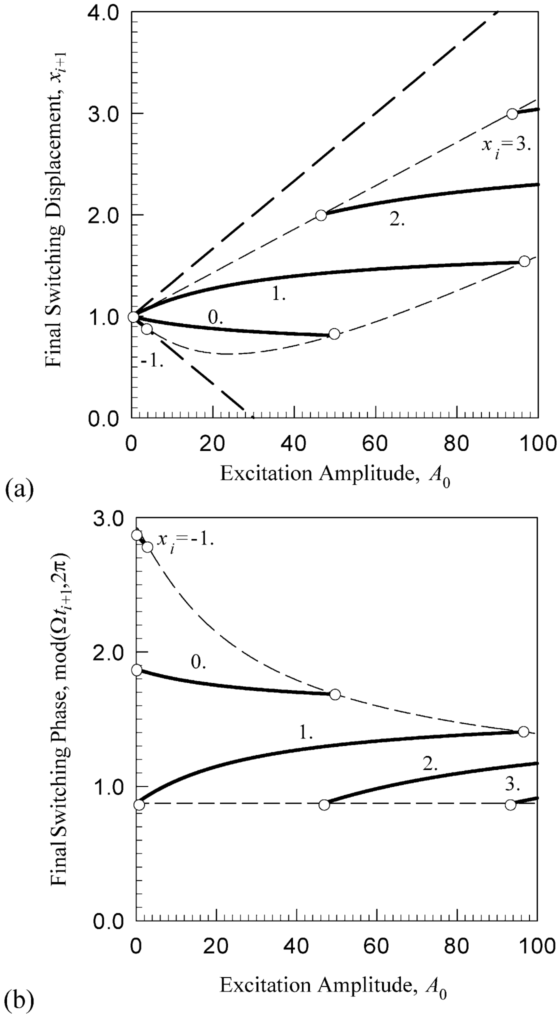
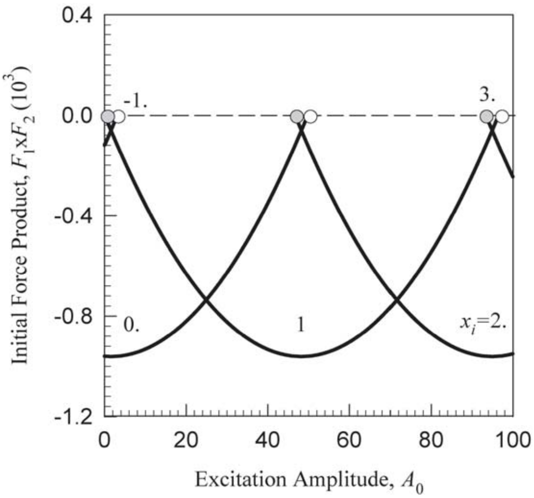
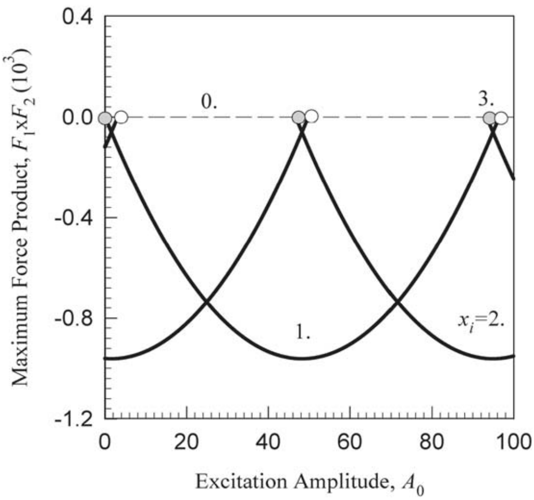


Figure 3.11. Effects of excitation amplitude for sliding motion vanishing on the boundary and going into the domain Ω_2: (a) final switching displacement, (b) final switching phase, (c) initial force product, and (d) maximum force product. ($V = 1, \Omega = 1, d_1 = 1, d_2 = 0, b_1 = -b_2 = 30, c_1 = c_2 = 30$, $\mathrm{mod}(\Omega t_i, 2\pi) \approx 0.8729$.)

Consider the basic system parameters $d_1 = 1, d_2 = 0, b_1 = -b_2 = b, c_1 = c_2 = 30$ for the sliding motion vanishing on the boundary and going to the domain Ω_2. The effects of the excitation amplitude and frequency to the final switching displacements and phases are presented in Fig. 3.11 for $V = 1, b =$



(c)



(d)

Figure 3.11. (*continued*)

30, $\mathrm{mod}(\Omega t_i, 2\pi) \approx 0.8729$, $\Omega = 1$. The final switching displacement versus the excitation amplitude are shown in Figs. 3.11(a) and (b). the final switching displacement versus the excitation amplitude is bounded in the right-hand-side area of the dark, dashed lines computed by Eq. (3.105). The thin, straight, dashed line boundary is the onset boundary determined by Eq. (3.102), and the curved boundary is determined by the initial force product condition in Eq. (3.92). Compared with the sliding motion relative to the domain Ω_1, no fragmentation and merging

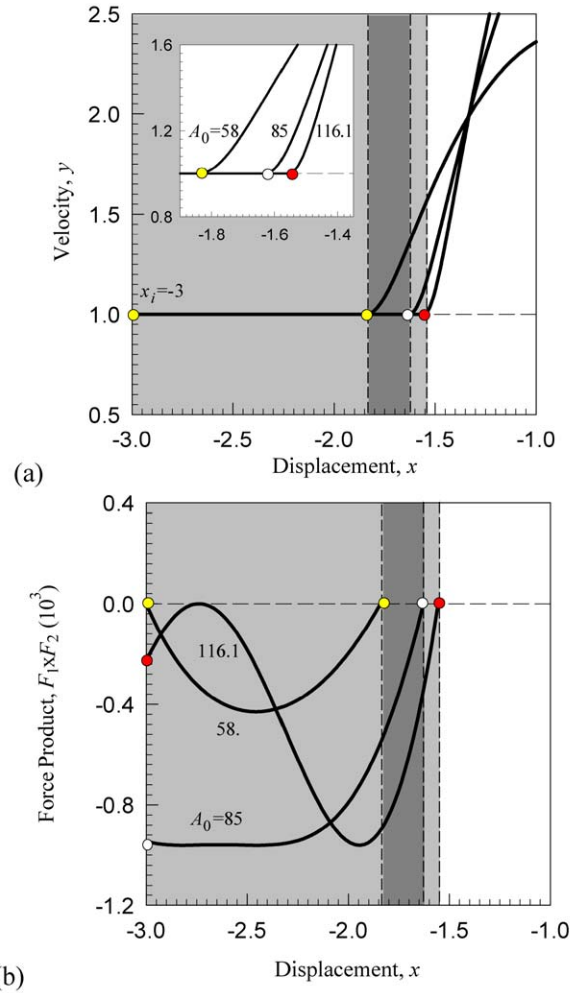
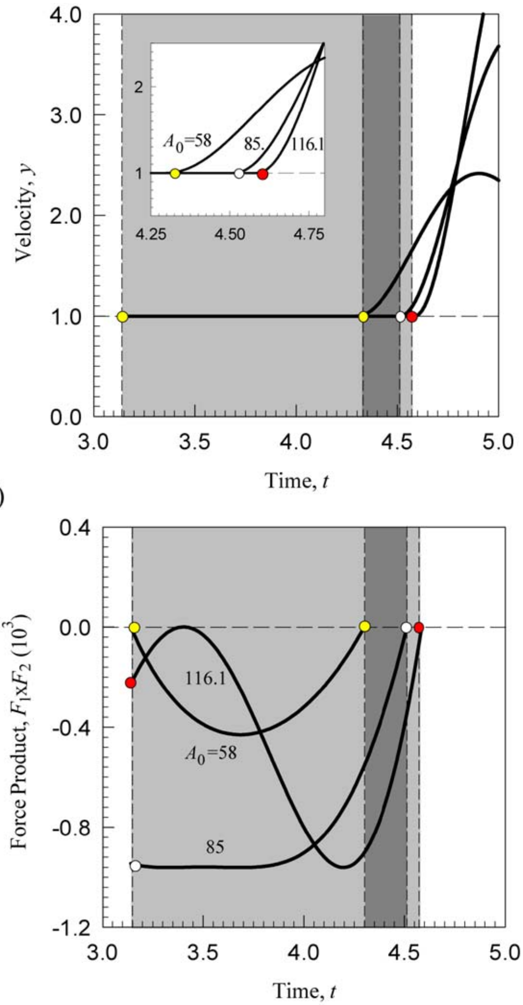


Figure 3.12. Sliding motion vanishing on the boundary and going into the domain Ω_1 for $A_0 = 58.0, 85.0, 116.1$: (a) phase plane, (b) force product versus displacement, (c) velocity time history, (d) force product time history. ($V = 1$, $\Omega = 1$, $d_1 = 1$, $d_2 = 0$, $b_1 = -b_2 = 30$, $c_1 = c_2 = 30$, $x_i = -3$, $\Omega t_i = \pi$.)

boundaries exists for this case. In an alike manner, the initial and maximum force products varying with excitation amplitude are presented in Figs. 3.11(c) and (d). The initial and maximum force products are identical. It indicates that no local maximum force product exists or the local maximum force product is smaller than the initial force product in the specific parameter range of excitation ampli-



(c)

(d)

Figure 3.12. (*continued*)

tude. In the domain Ω_1, for a given initial displacement, it is observed that the initial and maximum force products are identical. However, for larger excitation amplitude, the local maximum force product is greater than the initial force product. For the sliding motion, the effects of other parameters and initial conditions can be referred to Luo and Gegg (2006d).

Consider the parameters (i.e., $V = 1, \Omega = 1, d_1 = 1, d_2 = 0, b_1 = -b_2 = 30, c_1 = c_2 = 30$) with initial condition $x_i = 3, y_i = 1$ and $\Omega t_i = \pi$ for demonstration. For specific excitation amplitudes $A_0 = 58.0, 85.0, 116.1$, the

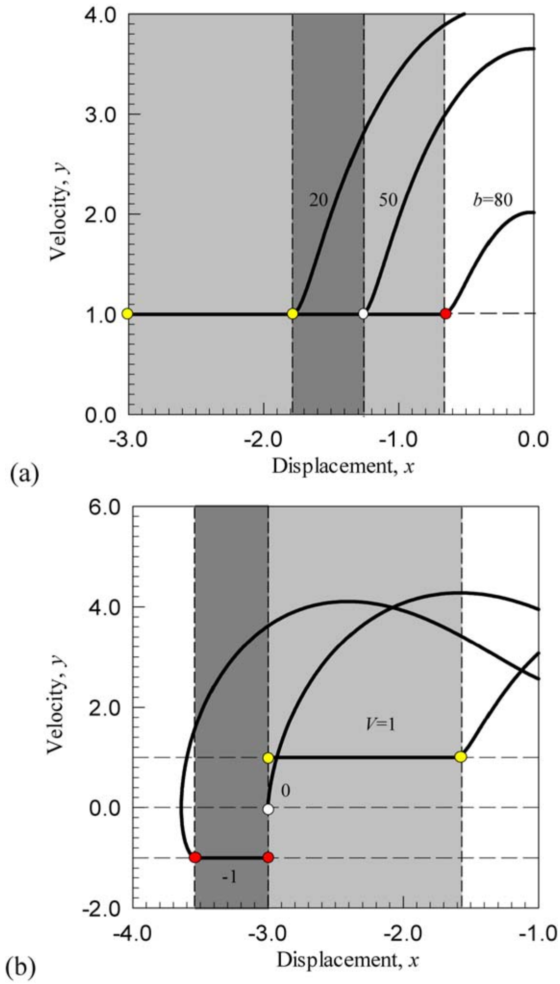
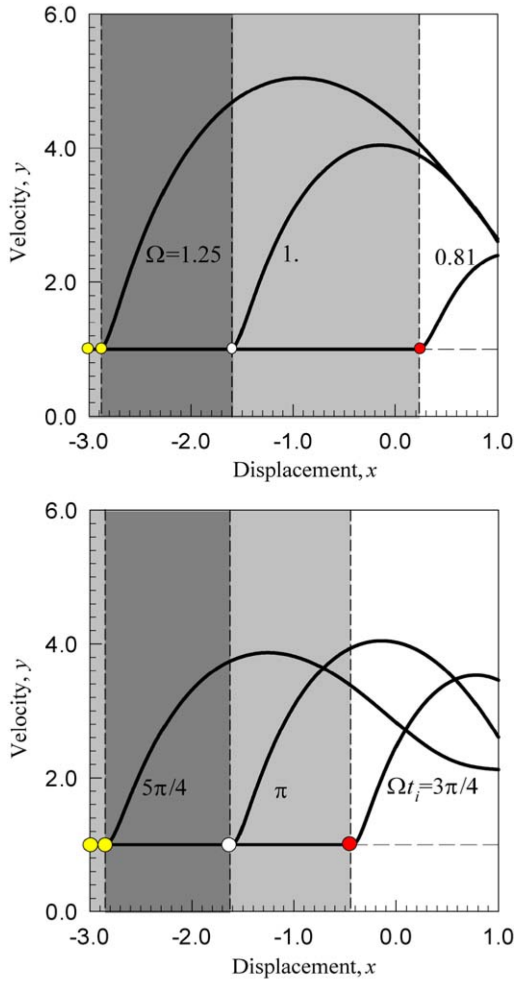


Figure 3.13. Phase planes of sliding motion disappearing on the boundary and going to the domain Ω_1: (a) $\Omega = 1, V = 1, \Omega t_i = \pi, b = 20, 50, 80$; (b) $\Omega = 1, b = 30, \Omega t_i = \pi, V = -1, 0, 1$; (c) $V = 1, b = 30, \Omega t_i = \pi, \Omega = 0.81, 1.0, 1.25$; (d) $\Omega = 1, V = 1, b = 30, \Omega t_i = 3\pi/4, \pi, 5\pi/4$. ($A_0 = 90, x_i = -3.0, d_1 = 1, d_2 = 0, b_1 = -b_2 = 30, c_1 = c_2 = 30$.)

phase trajectories of corresponding sliding motions are plotted in Fig. 3.12(a). The force product versus displacement for the sliding motion are presented in Fig. 3.12(b), and it is observed that the sliding motions satisfy $F_1 \times F_2 \leqslant 0$. For $A_0 = 58.0$, the initial force product is zero, it implies that $A_{0\,\min} \approx 58.0$. If $A_0 < A_{0\,\min}$, such a sliding motion cannot be observed. For $A_0 \approx 116.1$, the



(c)

(d)

Figure 3.13. (*continued*)

maximum force product is tangential zero. This value will be the maximum value for this sliding motion $A_{0\,\max} \approx 116.1$. If $A_0 > A_{0\,\max}$, the sliding motion will be fragmentized into two parts, as discussed in Fig. 3.7. Thus, under the above parameters, the sliding motion starting at $(x_i, y_i, \Omega t_i) = (-3.0, 1.0, \pi)$ will exist for $A_{0\,\min} \leqslant A_0 \leqslant A_{0\,\max}$. Further, the velocity and force product time-histories are also presented in Figs. 3.12(c) and (d). It is interesting that the force product time history is similar to the force product versus the displacement. The equiva-

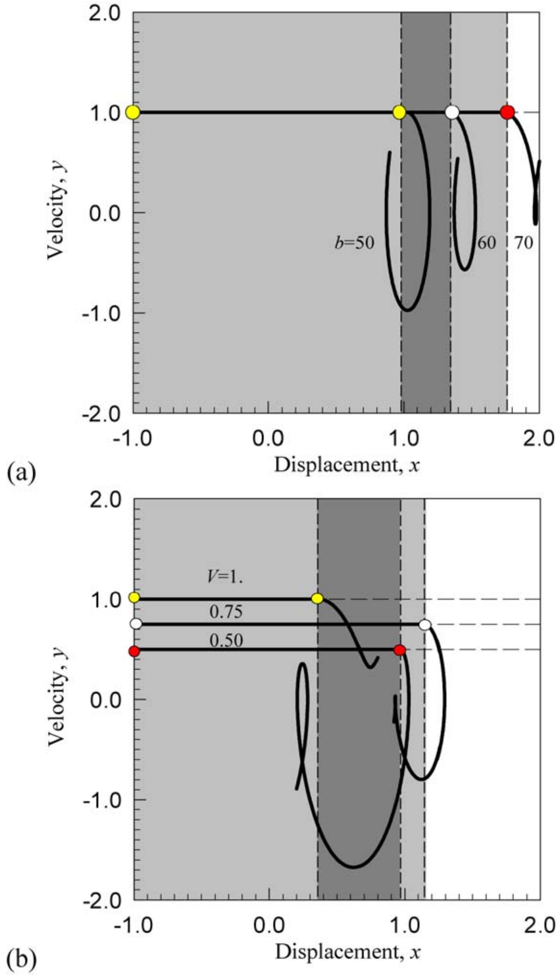
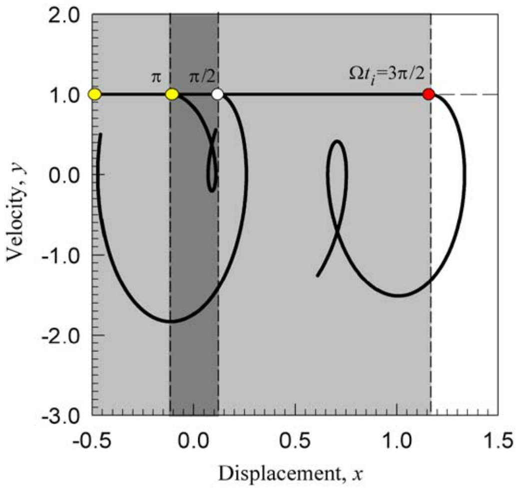
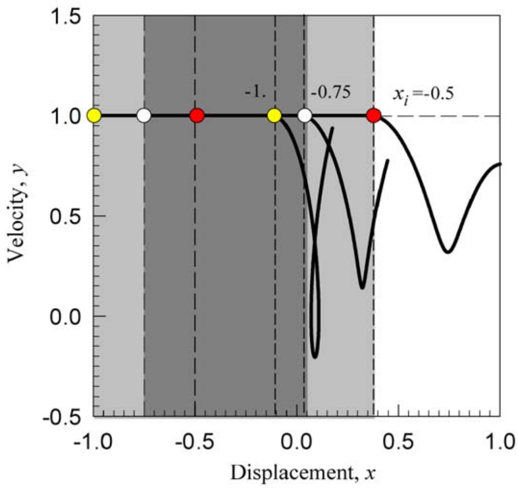


Figure 3.14. Phase planes of sliding motion disappearing on the boundary and going to the domain Ω_2: (a) $V = 1, x_i = -1, \Omega t_i \approx 0.8729, b = 50, 60, 70$; (b) $b = 15, x_i = -1, \Omega t_i = \pi$, $V = 0.5, 0.75, 1$; (c) $b = 15, V = 1, x_i = 3, \Omega t_i = \pi/2, \pi, 3\pi/2$; (d) $b = 15, V = 1$, $\Omega t_i = \pi, x_i = -1.0, -0.75, -0.5$. ($\Omega = 1, A_0 = 20, d_1 = 1, d_2 = 0, b_1 = -b_2 = b$, $c_1 = c_2 = 30$.)

lency of definitions in Eqs. (3.96) and (3.98) is verified numerically for nonzero constant belt speed. The phase trajectories of the sliding motions disappearing on the boundary and going to the domain Ω_1 are presented in Fig. 3.13 for parameters $A_0 = 90, x_i = -3, d_1 = 1, d_2 = 0, b_1 = -b_2 = b, c_1 = c_2 = 30$.



(c)



(d)

Figure 3.14. (*continued*)

In Fig. 3.12(a), the parameters $\Omega = 1, V = 1, \Omega t_i = \pi, b = 20, 50, 80$ are used. The parameters ($\Omega = 1, b = 30, \Omega t_i = \pi, V = -1, 0, 1$), ($V = 1, b = 30, \Omega t_i = \pi, \Omega = 0.81, 1.0, 1.25$) and ($\Omega = 1, b = 30, V = 1, \Omega t_i = 3\pi/4, \pi, 5\pi/4$) are also used in Figs. 3.13(b), (c) and (d), respectively. In Fig. 3.14, the phase trajectories of the sliding motion disappearing on the boundary and going to the domain Ω_2 are plotted through the parameters $\Omega = 1, A_0 = 20, d_1 = 1, d_2 = 0, b_1 = -b_2 = b, c_1 = c_2 = 30$. The four sets of the other parameters and initial conditions ($V = 1$, $x_i = -1$, $\Omega t_i \approx 0.8729$, $b = 50, 60, 70$), ($b = 15$, $x_i = -1$, $\Omega t_i = \pi$, $V = 0.5, 0.75, 1$), ($b = 15$, $V = 1$, $x_i = 3$, $\Omega t_i = \pi/2, \pi, 3\pi/2$) and ($b = 15$, $V = 1$, $\Omega t_i = \pi$, $x_i = -1.0, -0.75, -0.5$) are used in Figs. 3.14(a), (b), (c) and (d), respectively. From the above demonstrations of the phase trajectories, the analytical prediction and numerical simulations are in a very good agreement.

Chapter 4

Transversal Singularity and Bouncing Flows

The flow tangential to the separation boundary in an accessible domain was discussed in Chapter 2. The sliding flows on the boundary and the switching bifurcation between the passable and nonpassable flows on the boundary were discussed. In this chapter, the transversal tangential singularity and bouncing flows in the vicinity of the discontinuous boundary will be discussed. The basic concepts for such transversal tangential singularity will be introduced. The corresponding necessary and sufficient conditions for the transversal tangential flows on the passable separation boundaries are developed. The bouncing flows on the boundary will be discussed also. This phenomenon can be observed in dynamical systems with discontinuity from certain controls. A simple discontinuous dynamical system is presented herein for helping us understand such concepts.

4.1. Transversal tangential flows

Consider a flow $\mathbf{x}^{(\alpha)}(t)$ ($\alpha \in \{i, j\}$) in the vicinity of the boundary $\partial\Omega_{ij}$ with $\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t) > 0$ in the domain Ω_α for $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta$ for $t \in [t_{m-\varepsilon}, t_m)$, as shown in Fig. 2.4. In the domain Ω_β ($\alpha \neq \beta$), the flow $\mathbf{x}^{(\beta)}(t)$ in the vicinity of the boundary $\partial\Omega_{ij}$ possesses the condition $\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}(t) > 0$ at the same time interval. Therefore, the product of the normal components of the two flows satisfies the condition in Eq. (2.9), i.e., $\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-})\}\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+})\} > 0$. From Theorem 2.1, such a condition implies that the flow for the boundary $\partial\Omega_{ij}$ is passable. Therefore, the flow can pass through the boundary into another domain. The tangential bifurcation for a flow tangential to the separatrix was discussed without transversality in Chapter 2. The sliding dynamics along the discontinuous boundary and the switching bifurcation of flows on the boundary were discussed in Chapter 3. The tangency of the flow occurs just after the flow passes through the separation boundary. This tangential flow is termed *the transversal tangential flow*, and the mathematical definition is given as

DEFINITION 4.1. For a discontinuous dynamical system in Eq. (2.1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. Suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}^{(\beta)}(t_{m+})$ with $\alpha, \beta \in \{i, j\}$ $(\alpha \neq \beta)$ for $\mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}_m$. Both $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^{r-1}_{[t_{m-\varepsilon}, t_m)}$- and $C^{r}_{(t_m, t_{m+\varepsilon}]}$-continuous $(r \geqslant 2)$, respectively. The transverse flow $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}^{(0)}_{\alpha\beta}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ on the semi-passable boundary $\overrightarrow{\partial\Omega}_{\alpha\beta}$ is termed *a transversal tangential flow of the first kind* if the following conditions are satisfied:

$$\begin{aligned}
&\text{either} \quad \begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] > 0 \end{cases} \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta \\
&\text{or} \quad \begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] < 0 \end{cases} \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha;
\end{aligned} \tag{4.1}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+}) = 0; \tag{4.2}$$

$$\begin{aligned}
&\text{either} \quad \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m+})\right] > 0 \quad \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} > 0 \\
&\text{or} \quad \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m+})\right] < 0 \quad \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} < 0.
\end{aligned} \tag{4.3}$$

From the above definition, the first two condition gives a necessary condition of a flow tangential to the semi-passable boundary just after the flow passes through the boundary. The third condition determines the direction of a flow after the flow passes over the boundary, which is very strongly influenced by the sliding flow along the separation boundary. The direction of the component of the transversal tangential flow on the tangential vector of the separatrix $\partial\Omega_{ij}$ has the same direction of the sliding motion along the separatrix. Therefore, the third condition can be re-written as

$$\left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m+})\right] \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} > 0. \tag{4.4}$$

However, the computation of Eq. (4.3) is much easier and more intuitive than Eq. (4.4) because the flow $\mathbf{x}^{(0)}_{\alpha\beta}$ is determined by $\dot{\mathbf{x}}^{(0)}_{\alpha\beta} = \mathbf{F}^{(0)}_{\alpha\beta}(\mathbf{x}, t)$ for sliding dynamics in Chapter 3.

To illustrate the above concept, the geometrical description of flows passing through a boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$ are presented in Figs. 4.1 and 4.2. A pre-transversal tangential flow and a transversal tangential flow are sketched in Fig. 4.1. The tangency of the transverse flow occurs at the portion of the outflow. Consider a sliding motion along the positive direction of $\mathbf{t}_{\partial\Omega_{ij}}$ (i.e., $\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(0)}_{ij} > 0$). The pre-transversal tangential flow is a regular transverse flow

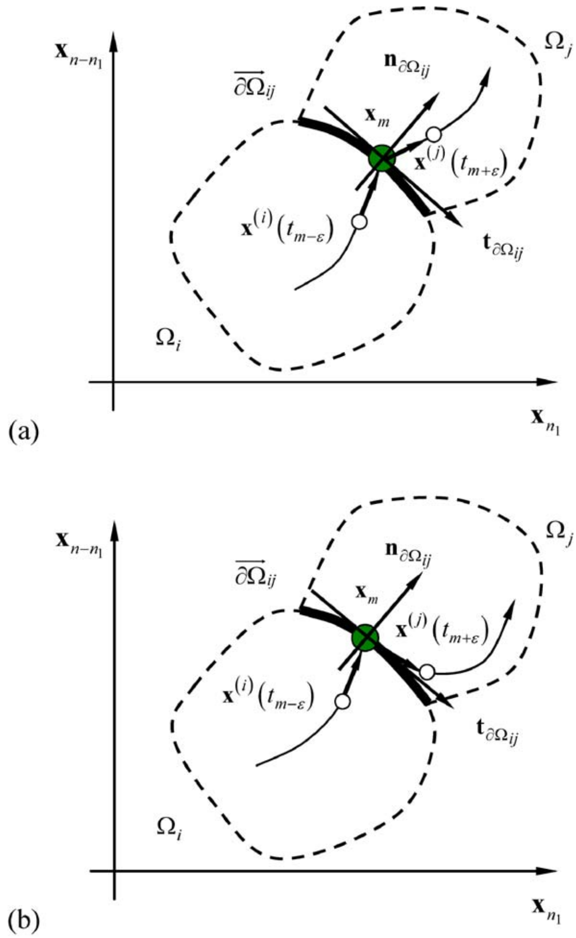


Figure 4.1. The flows of the pre-transversal tangential bifurcation to a boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$: (a) a pre-transversal tangential flow, and (b) a transversal tangential flow. The sliding motion is along the positive $\mathbf{t}_{\partial\Omega_{ij}}$ (i.e., $\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(0)}_{ij} > 0$).

from the domain Ω_i to Ω_j. The transversal tangential flow is a flow tangential to the separation boundary just after the flow passes over the boundary. After the transversal tangential bifurcation, a post-transversal tangential flow exists. For a post-transversal tangential flow, there are two intersected points on the separation boundary locally and the bouncing motion at the first intersected point will appear, as shown in Fig. 4.2(a). Because this tangential bifurcation may cause the bouncing motion, such a bifurcation can also termed "*the bouncing bifurca-*

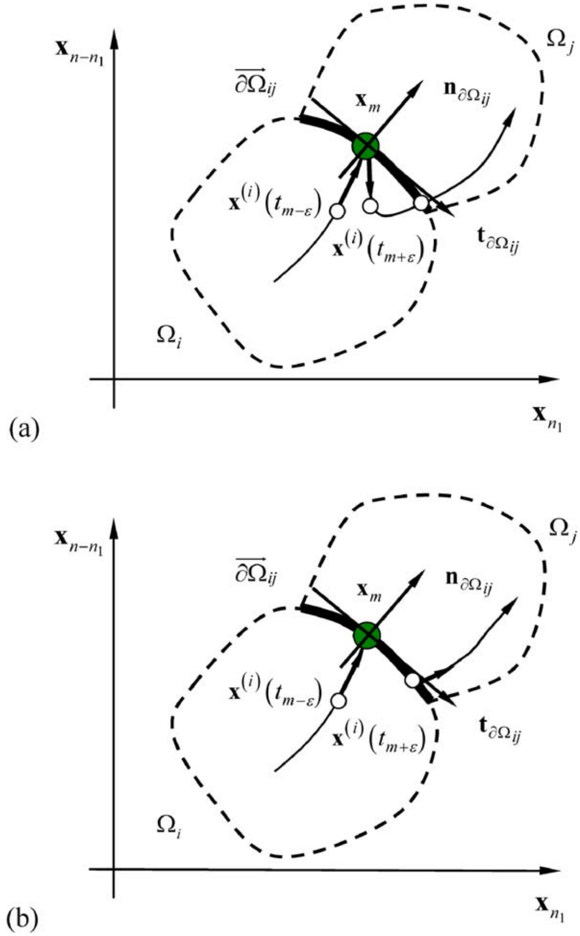


Figure 4.2. The flows of the post-transversal tangential bifurcation to a boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$: (a) a bouncing flow, and (b) sliding flow. The sliding motion is along the positive $\mathbf{t}_{\partial\Omega_{ij}}$ (i.e., $\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(0)}_{ij} > 0$).

tion". The post-transversal tangential bifurcation may cause the sliding motion in Fig. 4.2(b). Thus, this bifurcation can be termed "*the sliding bifurcation*", as discussed in Chapter 3.

THEOREM 4.1. *For a discontinuous dynamical system in Eq. (2.1),* $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$. *Suppose* $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}^{(\beta)}(t_{m+})$ *with* $\alpha, \beta \in \{i, j\}$

$(\alpha \neq \beta)$ for $\mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}_m$. Both $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^{r-1}_{[t_{m-\varepsilon},t_m)}$- and $C^{r}_{(t_m,t_{m+\varepsilon}]}$-continuous $(r \geqslant 2)$ respectively with $\|\mathrm{d}^r\mathbf{x}^{(\gamma)}/\mathrm{d}t^r\| < \infty$ $(\gamma \in \{\alpha,\beta\})$. The flow $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}^{(0)}_{\alpha\beta}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ on the semi-passable boundary $\overrightarrow{\partial\Omega}_{\alpha\beta}$ is a transversal tangential flow of the first kind iff

$$\begin{aligned} &\textit{either} \quad \left\{ \begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) > 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\dot{\mathbf{x}}^{(\beta)}(t_{m+\varepsilon}) - \dot{\mathbf{x}}^{(0)}(t_{m+\varepsilon})\right] > 0 \end{array} \right. \quad \textit{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta, \\ &\textit{or} \quad \left\{ \begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) < 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\dot{\mathbf{x}}^{(\beta)}(t_{m+\varepsilon}) - \dot{\mathbf{x}}^{(0)}(t_{m+\varepsilon})\right] < 0 \end{array} \right. \quad \textit{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha; \end{aligned} \tag{4.5}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+}) = 0; \tag{4.6}$$

$$\begin{aligned} &\textit{either} \quad \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+\varepsilon}) > 0 \quad \textit{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} > 0 \\ &\textit{or} \quad \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+\varepsilon}) < 0 \quad \textit{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} < 0. \end{aligned} \tag{4.7}$$

PROOF. For a point $\mathbf{x}^{(0)}_{\alpha\beta} \in \partial\Omega_{\alpha\beta}$ with $\mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta$, suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}^{(0)}_{\alpha\beta}$ and $\mathbf{x}^{(0)}_{\alpha\beta} = \mathbf{x}^{(\beta)}(t_{m+})$ with $\alpha,\beta \in \{i,j\}$ $(\alpha \neq \beta)$ for $\mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}_m$. Both $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^{r-1}_{[t_{m-\varepsilon},t_m)}$- and $C^{r}_{(t_m,t_{m+\varepsilon}]}$-continuous $(r \geqslant 2)$, respectively with $\|\mathrm{d}^r\mathbf{x}^{(\gamma)}/\mathrm{d}t^r\| < \infty$ $(\gamma \in \{\alpha,\beta\})$ for $0 < \varepsilon \ll 1$. Similarly to the proof of Theorem 2.1, application of the Taylor series expansion of $\mathbf{x}^{(\alpha)}(t_{m-\varepsilon})$ and $\mathbf{x}^{(\beta)}(t_{m+\varepsilon})$ with $t_{m\pm\varepsilon} = t_m \pm \varepsilon$ $(\alpha \in \{i,j\})$ to $\mathbf{x}^{(\alpha)}(t_{m\pm})$ and up to the second order term with $a \in [t_{m-\varepsilon}, t_{m-})$ gives

$$\begin{aligned} \mathbf{x}^{(\alpha)}(t_{m-\varepsilon}) &\equiv \mathbf{x}^{(\alpha)}(t_{m-} - \varepsilon) \\ &= \mathbf{x}^{(\alpha)}(a) + \dot{\mathbf{x}}^{(\alpha)}(a)(t_{m-} - a - \varepsilon) + o(t_{m-} - a - \varepsilon), \\ \mathbf{x}^{(\beta)}(t_{m+}) &\equiv \mathbf{x}^{(\beta)}(t_{m+\varepsilon} - \varepsilon) = \mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \dot{\mathbf{x}}^{(\beta)}(t_{m+\varepsilon})\varepsilon + o(\varepsilon). \end{aligned}$$

Let $a \to t_{m-}$. Because of $0 < \varepsilon \ll 1$, the second and higher order terms of the Taylor series expansion can be ignored in the foregoing equations. Therefore, we have

$$\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-})\varepsilon > 0 \quad \text{and} \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m+})\right] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+\varepsilon})\varepsilon > 0; \end{aligned}$$

Similarly,

$$\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(0)}(t_{m-}) - \mathbf{x}^{(0)}(t_{m-\varepsilon})\right] = \varepsilon\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}(t_{m-}) = 0, \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(0)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+})\right] = \varepsilon\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}(t_{m+\varepsilon}). \end{aligned}$$

Therefore,

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] = \varepsilon \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) > 0,$$
$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] = \varepsilon \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\dot{\mathbf{x}}^{(\beta)}(t_{m+\varepsilon}) - \dot{\mathbf{x}}^{(0)}(t_{m+\varepsilon})\right] > 0.$$

$$\begin{aligned}&\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m+})\right] \\ &\quad = \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+\varepsilon})\varepsilon \begin{cases} > 0 & \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} > 0, \\ < 0 & \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} < 0. \end{cases}\end{aligned}$$

From Definitions 2.8 and 4.1, the transverse flow $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}^{(0)}_{\alpha\beta}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ on the semi-passable boundary $\overrightarrow{\partial\Omega}_{\alpha\beta}$ is a transverse, tangential flow of the first kind for the boundary $\partial\Omega_{\alpha\beta}$ with $\mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta$. In a similar fashion, for the boundary $\partial\Omega_{\alpha\beta}$ with $\mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha$, it can be proved that the transverse flow $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}^{(0)}_{\alpha\beta}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ on the semi-passable boundary $\overrightarrow{\partial\Omega}_{\alpha\beta}$ is a transversal tangential flow of the first kind. □

THEOREM 4.2. *For a discontinuous dynamical system in Eq. (2.1),* $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$. *Suppose* $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}^{(\beta)}(t_{m+})$ *with* $\alpha, \beta \in \{i, j\}$ $(\alpha \neq \beta)$ *for* $\mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}_m$. *Both* $\mathbf{F}^{(\alpha)}(t)$ *and* $\mathbf{F}^{(\beta)}(t)$ *are* $C^{r-1}_{[t_{m-\varepsilon}, t_m)}$*- and* $C^{r}_{(t_m, t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 1)$ *respectively with* $\|\mathrm{d}^{r+1}\mathbf{x}^{(\gamma)}/\mathrm{d}t^{r+1}\| < \infty$ $(\gamma \in \{\alpha, \beta\})$. *The flow* $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}^{(0)}_{\alpha\beta}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ *on the semi-passable boundary* $\overrightarrow{\partial\Omega}_{\alpha\beta}$ *is a transversal tangential flow of the first kind iff*

$$\begin{aligned}&\textit{either} \quad \begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-}) > 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{F}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{F}^{(0)}(t_{m+\varepsilon})\right] > 0 \end{cases} \textit{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta \\ &\textit{or} \quad \begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-}) < 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{F}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{F}^{(0)}(t_{m+\varepsilon})\right] < 0 \end{cases} \textit{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha;\end{aligned} \tag{4.8}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}(t_{m+}) = 0; \tag{4.9}$$

$$\begin{aligned}&\textit{either} \quad \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}(t_{m+\varepsilon}) > 0 \quad \textit{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(0)}_{\alpha\beta} > 0 \\ &\textit{or} \quad \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}(t_{m+\varepsilon}) < 0 \quad \textit{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(0)}_{\alpha\beta} < 0.\end{aligned} \tag{4.10}$$

PROOF. With Eq. (2.1) and $\dot{\mathbf{x}}^{(0)}_{\alpha\beta} = \mathbf{F}^{(0)}_{\alpha\beta}(\mathbf{x}, t)$, the inequalities of Eqs. (4.5)–(4.7) gives Eqs. (4.8)–(4.10). So, the theorem is proved. □

THEOREM 4.3. *For a discontinuous dynamical system in Eq.* (2.1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$. *Suppose* $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}^{(\beta)}(t_{m+})$ *with* $\alpha, \beta \in \{i, j\}$ $(\alpha \neq \beta)$ *for* $\mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}_m$. *Both* $\mathbf{x}^{(\alpha)}(t)$ *and* $\mathbf{x}^{(\beta)}(t)$ *are* $C^{r-1}_{[t_{m-\varepsilon}, t_m)}$*- and* $C^{r}_{(t_m, t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 3)$, *respectively, with* $\|\mathrm{d}^{r-1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r-1}\| < \infty$ *and* $\|\mathrm{d}^{r}\mathbf{x}^{(\beta)}/\mathrm{d}t^{r}\| < \infty$. *The transverse flow* $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}^{(0)}_{\alpha\beta}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ *on the semi-passable boundary* $\overrightarrow{\partial\Omega}_{\alpha\beta}$ *is a transversal tangential flow of the first kind iff*

$$\begin{aligned} &\text{either} \quad \begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) > 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\ddot{\mathbf{x}}^{(\beta)}(t_{m+}) - \ddot{\mathbf{x}}^{(0)}(t_{m+})\right] > 0 \end{cases} \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta \\ &\text{or} \quad \begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) < 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\ddot{\mathbf{x}}^{(\beta)}(t_{m+}) - \ddot{\mathbf{x}}^{(0)}(t_{m+})\right] < 0 \end{cases} \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha; \end{aligned} \tag{4.11}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+}) = 0; \tag{4.12}$$

$$\begin{aligned} &\text{either} \quad \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+}) > 0 \quad \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} > 0 \\ &\text{or} \quad \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+}) < 0 \quad \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} < 0. \end{aligned} \tag{4.13}$$

PROOF. For an arbitrarily small $\varepsilon > 0$, the procedure of Theorem 2.1 is used. The application of the Taylor series expansion of $\mathbf{x}^{(\alpha)}(t_{m-\varepsilon})$ to $\mathbf{x}^{(\alpha)}(t_{m-})$ up to the second term with $a \in [t_{m-\varepsilon}, t_{m-})$ and $\mathbf{x}^{(\beta)}(t_{m+\varepsilon})$ to $\mathbf{x}^{(\beta)}(t_{m+})$ up to the third term with $b \in (t_{m+}, t_{m+\varepsilon}]$ gives

$$\begin{aligned} \mathbf{x}^{(\alpha)}(t_{m-\varepsilon}) &\equiv \mathbf{x}^{(\alpha)}(t_{m-} - \varepsilon) \\ &= \mathbf{x}^{(\alpha)}(a) + \dot{\mathbf{x}}^{(\alpha)}(a)(t_{m-} - \varepsilon - a) + o(t_{m-} - \varepsilon - a), \\ \mathbf{x}^{(\beta)}(t_{m+\varepsilon}) &\equiv \mathbf{x}^{(\beta)}(t_{m+} + \varepsilon) = \mathbf{x}^{(\beta)}(b) + \dot{\mathbf{x}}^{(\beta)}(b)(t_{m+} + \varepsilon - b) \\ &\quad + \ddot{\mathbf{x}}^{(\beta)}(b)(t_{m+} - \varepsilon - b)^2 + o\big((t_{m+} + \varepsilon - b)^2\big). \end{aligned}$$

Let $a \to t_{m-}$ and $b \to t_{m+}$. Since $0 < \varepsilon \ll 1$, the higher order terms can be ignored. The deformation of the above equations and multiplication by $\mathbf{n}_{\partial\Omega_{\alpha\beta}}$ and $\mathbf{t}_{\partial\Omega_{\alpha\beta}}$ leads to

$$\begin{aligned} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] &= \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-})\varepsilon, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m+})\right] &= \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+})\varepsilon \\ &\quad + \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \ddot{\mathbf{x}}^{(\beta)}(t_{m+})\varepsilon^2, \end{aligned}$$

$$\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m+})\right] = \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+})\varepsilon.$$

Similarly, we have the same expressions for $\dot{\mathbf{x}}^{(0)} = \mathbf{F}^{(0)}(\mathbf{x}, t)$. With Eq. (4.12), we have

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\ddot{\mathbf{x}}^{(\beta)}(t_{m+}) - \ddot{\mathbf{x}}^{(0)}(t_{m+})\right]\varepsilon^2.$$

For the boundary $\partial\Omega_{\alpha\beta}$ with $\mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta$, using the first inequality equation of Eq. (4.11), the foregoing two equations lead to

$$\begin{aligned}
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-})\varepsilon > 0,\\
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\ddot{\mathbf{x}}^{(\beta)}(t_{m+}) - \ddot{\mathbf{x}}^{(0)}(t_{m+})\right]\varepsilon^2 > 0.
\end{aligned}$$

Similarly, for the boundary $\partial\Omega_{\alpha\beta}$ with $\mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha$, using the second inequality equation of Eq. (4.11), the foregoing two equations lead to

$$\begin{aligned}
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-})\varepsilon < 0,\\
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\ddot{\mathbf{x}}^{(\beta)}(t_{m+}) - \ddot{\mathbf{x}}^{(0)}(t_{m+})\right]\varepsilon^2 < 0.
\end{aligned}$$

From Eq. (4.13), we have

$$\begin{aligned}
&\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m+})\right]\\
&\quad = \begin{cases} \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+})\varepsilon > 0 & \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} > 0,\\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+})\varepsilon < 0 & \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} < 0. \end{cases}
\end{aligned}$$

Therefore from Definition 4.1, the flow $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}^{(0)}_{\alpha\beta}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ on the semi-passable boundary $\overrightarrow{\partial\Omega}_{\alpha\beta}$ is a transversal tangential flow of the first kind. □

THEOREM 4.4. *For a discontinuous dynamical system in Eq. (2.1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. Suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}^{(\beta)}(t_{m+})$ with $\alpha, \beta \in \{i, j\}$ $(\alpha \neq \beta)$ for $\mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}_m$. Both $\mathbf{F}^{(\alpha)}(t)$ and $\mathbf{F}^{(\beta)}(t)$ are $C^{r-1}_{[t_{m-\varepsilon}, t_m)}$- and $C^{r}_{(t_m, t_{m+\varepsilon}]}$-continuous $(r \geqslant 2)$, respectively, with $\|\mathrm{d}^r\mathbf{x}^{(\alpha)}/\mathrm{d}t^r\| < \infty$ and $\|\mathrm{d}^{r+1}\mathbf{x}^{(\beta)}/\mathrm{d}t^{r+1}\| < \infty$. If the following conditions are satisfied*:

$$\begin{aligned}
&\textit{either} \quad \begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-}) > 0,\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{DF}^{(\beta)}(t_{m+}) - \mathbf{DF}^{(0)}(t_{m+})\right] > 0 \end{cases} \quad \textit{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta \\
&\textit{or} \quad \begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-}) < 0,\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{DF}^{(\beta)}(t_{m+}) - \mathbf{DF}^{(0)}(t_{m+})\right] < 0 \end{cases} \quad \textit{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha;
\end{aligned} \tag{4.14}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}(t_{m+}) = 0; \tag{4.15}$$

$$\begin{aligned} &\textit{either} \quad \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}(t_{m+}) > 0 \quad \textit{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(0)}_{\alpha\beta} > 0, \\ &\textit{or} \quad \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}(t_{m+}) < 0 \quad \textit{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(0)}_{\alpha\beta} < 0, \end{aligned} \tag{4.16}$$

then the transverse flow $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}^{(0)}_{\alpha\beta}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ *on the semi-passable boundary* $\overrightarrow{\partial\Omega}_{\alpha\beta}$ *is a transversal tangential flow of the first kind.*

PROOF. Using Eq. (2.1) and equation of motion for sliding dynamics (i.e., $\dot{\mathbf{x}}^{(0)}_{\alpha\beta} = \mathbf{F}^{(0)}_{\alpha\beta}(\mathbf{x}, t)$), from Theorem 4.3, the above theorem is proved. □

DEFINITION 4.2. For a discontinuous dynamical system in Eq. (2.1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. Suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}^{(\beta)}(t_{m+})$ with $\alpha, \beta \in \{i, j\}$ $(\alpha \neq \beta)$ for $\mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}_m$. Both $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^{r-2l+2}_{[t_{m-\varepsilon}, t_m)}$- and $C^{r}_{(t_m, t_{m+\varepsilon}]}$-continuous $(r \geqslant 2l)$, respectively. The transverse flow $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}^{(0)}_{\alpha\beta}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ on the semi-passable boundary $\overrightarrow{\partial\Omega}_{\alpha\beta}$ is termed *a transversal*, $(2l-1)$*th-order tangential flow of the first kind* if the following conditions are satisfied:

$$\begin{aligned} &\text{either} \quad \begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] > 0 \end{cases} \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta \\ &\text{or} \quad \begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] < 0 \end{cases} \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha; \end{aligned} \tag{4.17}$$

$$\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[\mathbf{x}^{(\beta)}(t_{m+}) - \mathbf{x}^{(0)}(t_{m+})\right] = 0 \quad \text{for } k = 1, 2, \ldots, 2l-1; \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \frac{\mathrm{d}^{2l}}{\mathrm{d}t^{2l}}\left[\mathbf{x}^{(\beta)}(t_{m+}) - \mathbf{x}^{(0)}(t_{m+})\right] \neq 0; \end{aligned} \tag{4.18}$$

$$\begin{aligned} &\text{either} \quad \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m+})\right] > 0 \quad \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} > 0 \\ &\text{or} \quad \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m+})\right] < 0 \quad \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} < 0. \end{aligned} \tag{4.19}$$

THEOREM 4.5. *For a discontinuous dynamical system in Eq.* (2.1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$. *Suppose* $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}^{(\beta)}(t_{m+})$ *with*

$\alpha, \beta \in \{i, j\}$ ($\alpha \neq \beta$) for $\mathbf{x}_{\alpha\beta}^{(0)}(t_m) = \mathbf{x}_m$. Both $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C_{[t_{m-\varepsilon}, t_m)}^{r-2l+2}$- and $C_{(t_m, t_{m+\varepsilon}]}^{r}$-continuous ($r \geqslant 2l$), respectively, with $\|\mathrm{d}^r \mathbf{x}^{(\beta)}/\mathrm{d}t^r\| < \infty$ and $\|\mathrm{d}^{r-2l+2}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r-2l+2}\| < \infty$. The transverse flow $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}_{\alpha\beta}^{(0)}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ on the semi-passable boundary $\overrightarrow{\partial\Omega}_{\alpha\beta}$ a transversal, $(2l-1)$th-order tangential flow of the first kind iff

$$\begin{aligned} &\text{either} \quad \begin{cases} \mathbf{n}_{\partial\Omega_{\alpha\beta}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) > 0, \\ \mathbf{n}_{\partial\Omega_{\alpha\beta}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+\varepsilon}) > 0 \end{cases} \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta \\ &\text{or} \quad \begin{cases} \mathbf{n}_{\partial\Omega_{\alpha\beta}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) < 0, \\ \mathbf{n}_{\partial\Omega_{\alpha\beta}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+\varepsilon}) < 0 \end{cases} \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha; \end{aligned} \tag{4.20}$$

$$\begin{aligned} &\mathbf{n}_{\partial\Omega_{\alpha\beta}}^{\mathrm{T}} \cdot \frac{\mathrm{d}^k}{\mathrm{d}t^k}\big[\mathbf{x}^{(\beta)}(t_{m+}) - \mathbf{x}^{(0)}(t_{m+})\big] = 0 \quad \text{for } k = 1, 2, \ldots, 2l-1; \\ &\mathbf{n}_{\partial\Omega_{\alpha\beta}}^{\mathrm{T}} \cdot \frac{\mathrm{d}^{(2l)}}{\mathrm{d}t^{(2l)}}\big[\mathbf{x}^{(\beta)}(t_{m+}) - \mathbf{x}^{(0)}(t_{m+})\big] \neq 0; \end{aligned} \tag{4.21}$$

$$\begin{aligned} &\text{either} \quad \mathbf{t}_{\partial\Omega_{\alpha\beta}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+}) > 0 \quad \text{for } \mathbf{t}_{\partial\Omega_{\alpha\beta}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}_{\alpha\beta}^{(0)} > 0 \\ &\text{or} \quad \mathbf{t}_{\partial\Omega_{\alpha\beta}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+}) < 0 \quad \text{for } \mathbf{t}_{\partial\Omega_{\alpha\beta}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}_{\alpha\beta}^{(0)} < 0. \end{aligned} \tag{4.22}$$

PROOF. Following the proof procedure of Theorem 2.8, and using the Taylor series expansion, the above theorem can be proved. □

THEOREM 4.6. *For a discontinuous dynamical system in Eq. (2.1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. Suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}_{\alpha\beta}^{(0)}(t_m) = \mathbf{x}^{(\beta)}(t_{m+})$ with $\alpha, \beta \in \{i, j\}$ ($\alpha \neq \beta$) for $\mathbf{x}_{\alpha\beta}^{(0)}(t_m) = \mathbf{x}_m$. Both $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C_{[t_{m-\varepsilon}, t_m)}^{r-2l+2}$- and $C_{(t_m, t_{m+\varepsilon}]}^{r}$-continuous ($r \geqslant 2l-1$), respectively. The transverse flow $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}_{\alpha\beta}^{(0)}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ on the semi-passable boundary $\overrightarrow{\partial\Omega}_{\alpha\beta}\overrightarrow{\partial\Omega}_{\alpha\beta}$ is a transversal, $(2l-1)$th-order tangential flow of the first kind iff*

$$\begin{aligned} &\text{either} \quad \begin{cases} \mathbf{n}_{\partial\Omega_{\alpha\beta}}^{\mathrm{T}} \cdot \mathbf{F}^{(\alpha)}(t_{m-}) > 0, \\ \mathbf{n}_{\partial\Omega_{\alpha\beta}}^{\mathrm{T}} \cdot \big[\mathbf{F}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{F}^{(0)}(t_{m+\varepsilon})\big] > 0 \end{cases} \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta, \\ &\text{or} \quad \begin{cases} \mathbf{n}_{\partial\Omega_{\alpha\beta}}^{\mathrm{T}} \cdot \mathbf{F}^{(\alpha)}(t_{m-}) < 0, \\ \mathbf{n}_{\partial\Omega_{\alpha\beta}}^{\mathrm{T}} \cdot \big[\mathbf{F}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{F}^{(0)}(t_{m+\varepsilon})\big] < 0 \end{cases} \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha; \end{aligned} \tag{4.23}$$

$$\mathbf{n}_{\partial\Omega_{\alpha\beta}}^{\mathrm{T}} \cdot \mathrm{D}^{k-1}\big[\mathbf{F}^{(\beta)}(t_{m+}) - \mathbf{F}^{(0)}(t_{m+})\big] = 0 \quad \text{for } k = 1, 2, \ldots, 2l-1, \tag{4.24}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathrm{D}^{2l-1}\big[\mathbf{F}^{(\beta)}(t_{m+}) - \mathbf{F}^{(0)}(t_{m+})\big] \neq 0;$$

$$\begin{aligned} &\textit{either} \quad \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}(t_{m+}) > 0 \quad \textit{for}\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(0)}_{\alpha\beta} > 0 \\ &\textit{or} \quad \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}(t_{m+}) < 0 \quad \textit{for}\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(0)}_{\alpha\beta} < 0. \end{aligned} \tag{4.25}$$

PROOF. Following the proof procedure of Theorem 2.9 and using of the Taylor series expansion, the above theorem can be proved. □

THEOREM 4.7. *For a discontinuous dynamical system in Eq.* (2.1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$. *Suppose* $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}^{(\beta)}(t_{m+})$ *with* $\alpha, \beta \in \{i, j\}$ $(\alpha \neq \beta)$ *for* $\mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}_m$. *Both* $\mathbf{x}^{(\alpha)}(t)$ *and* $\mathbf{x}^{(\beta)}(t)$ *are* $C^{r-2l+1}_{[t_{m-\varepsilon}, t_m)}$*- and* $C^{r}_{(t_m, t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 2l)$, *respectively, with* $\|\mathrm{d}^r\mathbf{x}^{(\beta)}/\mathrm{d}t^r\| < \infty$ *and* $\|\mathrm{d}^{r-2l+1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r-2l+1}\| < \infty$. *The transverse flow* $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}^{(0)}_{\alpha\beta}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ *on the semi-passable boundary* $\overrightarrow{\partial\Omega}_{\alpha\beta}$ *a transversal,* $(2l-1)$*th-order tangential flow of the first kind iff*

$$\begin{aligned} &\textit{either} \quad \left\{ \begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) > 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \frac{\mathrm{d}^{2l}}{\mathrm{d}t^{2l}}\big[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\big] > 0 \end{array} \right. \quad \textit{for}\ \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta \\ &\textit{or} \quad \left\{ \begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) < 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \frac{\mathrm{d}^{2l}}{\mathrm{d}t^{2l}}\big[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\big] < 0 \end{array} \right. \quad \textit{for}\ \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha; \end{aligned} \tag{4.26}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \frac{\mathrm{d}^k}{\mathrm{d}t^k}\mathbf{x}^{(\beta)}(t_{m+}) = 0 \quad (k = 1, 2, \ldots, 2l-1); \tag{4.27}$$

$$\begin{aligned} &\textit{either} \quad \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+}) > 0 \quad \textit{for}\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} > 0 \\ &\textit{or} \quad \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+}) < 0 \quad \textit{for}\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} < 0. \end{aligned} \tag{4.28}$$

PROOF. Following the proof procedure of Theorem 2.10 and using the Taylor series expansion, the above theorem can be proved. □

THEOREM 4.8. *For a discontinuous dynamical system in Eq.* (2.1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$. *Suppose* $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}^{(\beta)}(t_{m+})$ *with* $\alpha, \beta \in \{i, j\}$ $(\alpha \neq \beta)$ *for* $\mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}_m$. *Both* $\mathbf{F}^{(\alpha)}(t)$ *and* $\mathbf{F}^{(\beta)}(t)$ *are* $C^{r-2l+2}_{[t_{m-\varepsilon}, t_m)}$*-*

and $C^{r}_{(t_m,t_{m+\varepsilon}]}$*-continuous* ($r \geqslant 2l-1$), *respectively, with* $\|\mathrm{d}^{r+1}\mathbf{x}^{(\beta)}/\mathrm{d}t^{r+1}\| < \infty$ *and* $\|\mathrm{d}^{r-2l+3}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r-2l+3}\| < \infty$. *The transverse flow* $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}^{(0)}_{\alpha\beta}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ *on the semi-passable boundary* $\overrightarrow{\partial\Omega}_{\alpha\beta}$ *a transversal,* $(2l-1)$*th-order tangential flow of the first kind iff*

$$\begin{aligned}
&\text{either} \quad \left\{\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(t_{m-}) > 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathrm{D}^{2l-1}\big[\mathbf{F}^{(\beta)}(t_{m+}) - \mathbf{F}^{(0)}(t_{m+})\big] > 0 \end{array}\right. \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_{\beta} \\
&\text{or} \quad \left\{\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(t_{m-}) < 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathrm{D}^{2l-1}\big[\mathbf{F}^{(\beta)}(t_{m+}) - \mathbf{F}^{(0)}(t_{m+})\big] < 0 \end{array}\right. \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_{\alpha};
\end{aligned} \tag{4.29}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathrm{D}^{k-1}\big[\mathbf{F}^{(\beta)}(t_{m+}) - \mathbf{F}^{(0)}(t_{m+})\big] = 0 \quad (k = 1, 2, \ldots, 2l-1); \tag{4.30}$$

$$\begin{aligned}
&\text{either} \quad \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}(t_{m+}) > 0 \quad \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(0)}_{\alpha\beta} > 0 \\
&\text{or} \quad \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}(t_{m+}) < 0 \quad \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(0)}_{\alpha\beta} < 0.
\end{aligned} \tag{4.31}$$

PROOF. Following the proof procedure of Theorem 2.11 and using the Taylor series expansion, the above theorem can be proved. □

DEFINITION 4.3. For a discontinuous dynamical system in Eq. (2.1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. Suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}^{(\beta)}(t_{m+})$ with $\alpha, \beta \in \{i, j\}$ ($\alpha \neq \beta$). Both $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^{r-1}_{[t_{m-\varepsilon},t_m)}$ and $C^{r}_{(t_m,t_{m+\varepsilon}]}$-continuous ($r \geqslant 2$), respectively. The transverse flow $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}^{(0)}_{\alpha\beta}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ on the semi-passable boundary $\overrightarrow{\partial\Omega}_{\alpha\beta}$ is termed *a transversal tangential flow of the second kind* if the following conditions are satisfied:

$$\begin{aligned}
&\text{either} \quad \left\{\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \big[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\big] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \big[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\big] > 0 \end{array}\right. \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_{\beta} \\
&\text{or} \quad \left\{\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \big[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\big] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \big[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\big] < 0 \end{array}\right. \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_{\alpha},
\end{aligned} \tag{4.32}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) = 0. \tag{4.33}$$

If the conditions in relative to the β-term in Eq. (4.32) is replaced by

$$\text{either} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \big[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\big] > 0 \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_{\beta}$$

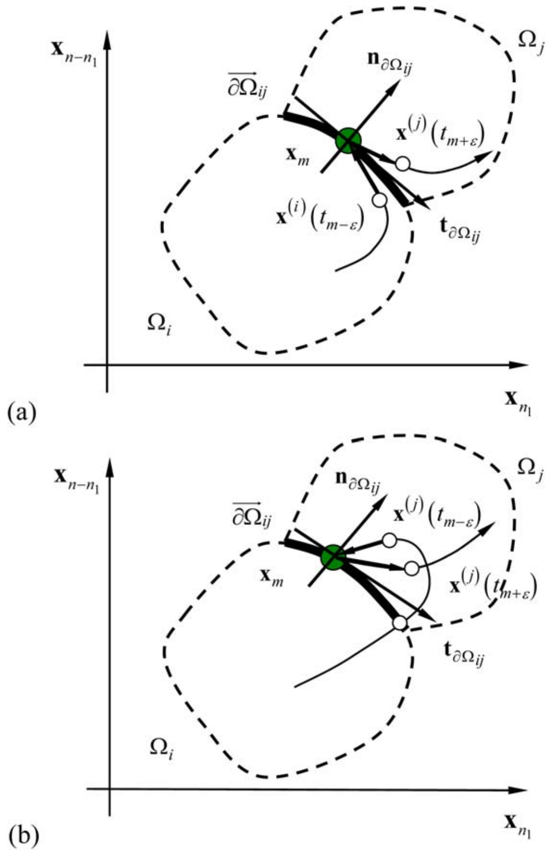


Figure 4.3. The flows passing through a boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$: (a) a transversal tangential flow of the second kind, and (b) a post-transversal tangential flow for $\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(0)}_{ij} > 0$.

$$\text{or} \qquad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] > 0 \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha. \tag{4.34}$$

The foregoing definition gives the grazing bifurcation, which was discussed in Chapter 2. If those conditions are replaced by

$$\begin{aligned} &\text{either} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] < 0 \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta \\ &\text{or} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] > 0 \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha, \end{aligned} \tag{4.35}$$

the sliding bifurcation will be defined. Similarly, a transversal, $(2l-1)$th-order tangential flow of the second kind will be introduced as follows.

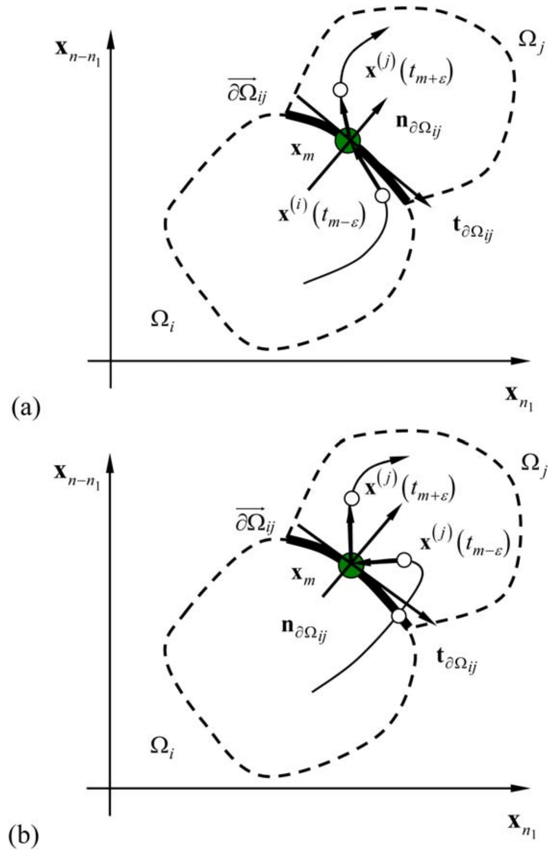


Figure 4.4. The flows passing through a boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$: (a) a transversal tangential flow of the second kind, and (b) a post-transversal tangential flow for $\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(0)}_{ij} < 0$.

DEFINITION 4.4. For a discontinuous dynamical system in Eq. (2.1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. Suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}^{(\beta)}(t_{m+})$ with $\alpha, \beta \in \{i, j\}$ $(\alpha \neq \beta)$ for $\mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}_m$. Both $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^{r-2l+2}_{[t_{m-\varepsilon}, t_m)}$- and $C^{r}_{(t_m, t_{m+\varepsilon}]}$-continuous $(r \geqslant 2l)$, respectively. The transverse flow $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}^{(0)}_{\alpha\beta}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ on the semi-passable boundary $\overrightarrow{\partial\Omega}_{\alpha\beta}$ is termed *a transversal*, $(2l-1)$*th-order tangential flow of the second kind* if the following conditions are satisfied:

$$\begin{aligned}&\text{either}\quad \begin{cases}\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\left[\mathbf{x}^{(0)}(t_{m-\varepsilon})-\mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right]>0\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon})-\mathbf{x}^{(0)}(t_{m+\varepsilon})\right]>0\end{cases}\quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}}\to\Omega_{\beta}\\ &\text{or}\quad \begin{cases}\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\left[\mathbf{x}^{(0)}(t_{m-\varepsilon})-\mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right]<0\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon})-\mathbf{x}^{(0)}(t_{m+\varepsilon})\right]<0\end{cases}\quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}}\to\Omega_{\alpha};\end{aligned}\tag{4.36}$$

$$\begin{aligned}&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[\mathbf{x}^{(\alpha)}(t_{m-})-\mathbf{x}^{(0)}(t_{m-})\right]=0\quad \text{for } k=1,2,\ldots,2n-1,\\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}}\left[\mathbf{x}^{(\alpha)}(t_{m-})-\mathbf{x}^{(0)}(t_{m-})\right]\neq 0.\end{aligned}\tag{4.37}$$

The theorems for the transversal tangential flows of the *second* kind on the semi-passable boundary $\overrightarrow{\partial\Omega}_{ij}$ can be similar to Theorems 4.1–4.8 for the transversal tangential flow of the *first* kind. The transversal tangential flows of the *second* kind on the semi-passable boundary $\overrightarrow{\partial\Omega}_{ij}$ are illustrated in Figs. 4.3 and 4.4. The flows crossing over the boundary for two cases ($\mathbf{t}_{\partial\Omega_{ij}}\cdot\dot{\mathbf{x}}^{(0)}_{ij}>0$ and $\mathbf{t}_{\partial\Omega_{ij}}\cdot\dot{\mathbf{x}}^{(0)}_{ij}<0$) are demonstrated. The transversality of flow on the boundary should be used. Compared to the transversal tangential flow of the first kind, the input tangential flow is *independent* of the sliding flow. After the transversal tangential bifurcation of the second kind occurs, the bouncing motion appears in the post-transversal tangential flow. The outflow of the bouncing motion is strongly dependent upon the sliding motion.

4.2. Cusped and inflexed tangential flows

The tangency of a flow occurs just before and after the flow passes through the separation boundary. This tangential flow includes cusped and inflexed tangential flows. The definitions are given as follows.

DEFINITION 4.5. For a discontinuous dynamical system in Eq. (2.1), $\mathbf{x}(t_m)\equiv\mathbf{x}_m\in\partial\Omega_{ij}$ for t_m. For an arbitrarily small $\varepsilon>0$, there are two time intervals $[t_{m-\varepsilon},t_m)$ and $(t_m,t_{m+\varepsilon}]$. Suppose $\mathbf{x}^{(\alpha)}(t_{m-})=\mathbf{x}^{(0)}_{\alpha\beta}(t_m)=\mathbf{x}^{(\beta)}(t_{m+})$ with $\alpha,\beta\in\{i,j\}$ $(\alpha\neq\beta)$ for $\mathbf{x}^{(0)}_{\alpha\beta}(t_m)=\mathbf{x}_m$. Both $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^{r-1}_{[t_{m-\varepsilon},t_m)}$- and $C^{r}_{(t_m,t_{m+\varepsilon}]}$-continuous $(r\geqslant 2)$, respectively. The transverse flow $\mathbf{x}=\mathbf{x}^{(\alpha)}(t<t_m)\cup\mathbf{x}^{(0)}_{\alpha\beta}(t_m)\cup\mathbf{x}^{(\beta)}(t>t_m)$ passing through $\overrightarrow{\partial\Omega}_{\alpha\beta}$ is termed *a transversal, cusped tangential flow* if the following conditions are satisfied:

$$\text{either}\quad \begin{cases}\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\left[\mathbf{x}^{(0)}(t_{m-\varepsilon})-\mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right]>0\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon})-\mathbf{x}^{(0)}(t_{m+\varepsilon})\right]>0\end{cases}\quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}}\to\Omega_{\beta}\tag{4.38}$$

$$\text{or} \quad \begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] < 0 \end{cases} \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_{\alpha};$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+}) = 0; \tag{4.39}$$

$$\begin{aligned} &\text{either} \quad \begin{cases} \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] < 0 \\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] > 0 \end{cases} \quad \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} > 0 \\ &\text{or} \quad \begin{cases} \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] > 0 \\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] < 0 \end{cases} \quad \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} < 0. \end{aligned} \tag{4.40}$$

DEFINITION 4.6. For a discontinuous dynamical system in Eq. (2.1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. Suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}^{(\beta)}(t_{m+})$ with $\alpha, \beta \in \{i, j\}$ $(\alpha \neq \beta)$ for $\mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}_m$. Both $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^{r-1}_{[t_{m-\varepsilon}, t_m)}$- and $C^{r}_{(t_m, t_{m+\varepsilon}]}$-continuous $(r \geqslant 2)$, respectively. The transverse flow $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}^{(0)}_{\alpha\beta}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ passing through $\overrightarrow{\partial\Omega}_{\alpha\beta}$ is termed the transversal, inflexed tangential flow if the following conditions are satisfied:

$$\begin{aligned} &\text{either} \quad \begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] > 0 \end{cases} \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_{\beta} \\ &\text{or} \quad \begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] < 0 \end{cases} \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_{\alpha}; \end{aligned} \tag{4.41}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+}) = 0; \tag{4.42}$$

$$\begin{aligned} &\text{either} \quad \begin{cases} \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] > 0 \\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] > 0 \end{cases} \quad \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} > 0 \\ &\text{or} \quad \begin{cases} \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] < 0 \\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] < 0 \end{cases} \quad \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} < 0. \end{aligned} \tag{4.43}$$

DEFINITION 4.7. For a discontinuous dynamical system in Eq. (2.1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. Suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}^{(\beta)}(t_{m+})$ with $\alpha, \beta \in \{i, j\}$ $(\alpha \neq \beta)$ for $\mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}_m$. Both $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^{r-2q+2p}_{[t_{m-\varepsilon}, t_m)}$- and $C^{r}_{(t_m, t_{m+\varepsilon}]}$-continuous $(r \geqslant 2q)$, respectively. The transverse flow $\mathbf{x} = \mathbf{x}^{(\alpha)}(t <$

$t_m) \cup \mathbf{x}^{(0)}_{\alpha\beta}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ passing through $\overrightarrow{\partial\Omega}_{\alpha\beta}$ is termed the transversal, cusped, $(2p-1:2q-1)$-order tangential flow if the following conditions are satisfied:

$$\begin{aligned}&\text{either} \quad \begin{cases}\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\left[\mathbf{x}^{(0)}(t_{m-\varepsilon})-\mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right]>0\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon})-\mathbf{x}^{(0)}(t_{m+\varepsilon})\right]>0\end{cases} \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}}\to\Omega_\beta\\ &\text{or} \quad \begin{cases}\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\left[\mathbf{x}^{(0)}(t_{m-\varepsilon})-\mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right]<0\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon})-\mathbf{x}^{(0)}(t_{m+\varepsilon})\right]<0\end{cases} \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}}\to\Omega_\alpha,\end{aligned} \tag{4.44}$$

$$\begin{aligned}&\begin{cases}\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\frac{\mathrm{d}^{k_1}}{\mathrm{d}t^{k_1}}\left[\mathbf{x}^{(\alpha)}(t_{m-})-\mathbf{x}^{(0)}(t_{m-})\right]=0 \quad (k_1=1,2,\ldots,2p-1),\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\frac{\mathrm{d}^{2p}}{\mathrm{d}t^{2q}}\left[\mathbf{x}^{(\alpha)}(t_{m-})-\mathbf{x}^{(0)}(t_{m-})\right]\neq 0,\end{cases}\\ &\begin{cases}\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\frac{\mathrm{d}^{k_2}}{\mathrm{d}t^{k_2}}\left[\dot{\mathbf{x}}^{(\beta)}(t_{m+})-\dot{\mathbf{x}}^{(0)}(t_{m+})\right]=0 \quad (k_2=1,2,\ldots,2q-1),\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\frac{\mathrm{d}^{2p}}{\mathrm{d}t^{2q}}\left[\dot{\mathbf{x}}^{(\beta)}(t_{m+})-\dot{\mathbf{x}}^{(0)}(t_{m+})\right]\neq 0;\end{cases}\end{aligned} \tag{4.45}$$

$$\begin{aligned}&\text{either} \quad \begin{cases}\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\left[\mathbf{x}^{(\alpha)}(t_{m-})-\mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right]<0\\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon})-\mathbf{x}^{(\beta)}(t_{m+})\right]>0\end{cases} \quad \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(0)}_{\alpha\beta}>0\\ &\text{or} \quad \begin{cases}\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\left[\mathbf{x}^{(\alpha)}(t_{m-})-\mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right]>0\\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon})-\mathbf{x}^{(\beta)}(t_{m+})\right]<0\end{cases} \quad \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(0)}_{\alpha\beta}<0.\end{aligned} \tag{4.46}$$

DEFINITION 4.8. For a discontinuous dynamical system in Eq. (2.1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. Suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}^{(\beta)}(t_{m+})$ with $\alpha, \beta \in \{i, j\}$ $(\alpha \neq \beta)$ for $\mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}_m$. Both $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^{r-2q+2p}_{[t_{m-\varepsilon},t_m)}$- and $C^r_{(t_m,t_{m+\varepsilon}]}$-continuous $(r \geqslant 2q)$, respectively. The transverse flow $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}^{(0)}_{\alpha\beta}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ passing through $\overrightarrow{\partial\Omega}_{\alpha\beta}$ is termed the transversal, inflexed, $(2p-1:2q-1)$-order tangential flow if the following conditions are satisfied:

$$\begin{aligned}&\text{either} \quad \begin{cases}\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\left[\mathbf{x}^{(0)}(t_{m-\varepsilon})-\mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right]>0\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon})-\mathbf{x}^{(0)}(t_{m+\varepsilon})\right]>0\end{cases} \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}}\to\Omega_\beta\\ &\text{or} \quad \begin{cases}\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\left[\mathbf{x}^{(0)}(t_{m-\varepsilon})-\mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right]<0\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon})-\mathbf{x}^{(0)}(t_{m+\varepsilon})\right]<0\end{cases} \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}}\to\Omega_\alpha\end{aligned} \tag{4.47}$$

and

$$\begin{cases}\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\frac{\mathrm{d}^{k_1}}{\mathrm{d}t^{k_1}}\left[\mathbf{x}^{(\alpha)}(t_{m-})-\mathbf{x}^{(0)}(t_{m-})\right]=0 \quad (k_1=1,2,\ldots,2p-1),\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\frac{\mathrm{d}^{2p}}{\mathrm{d}t^{2p}}\left[\mathbf{x}^{(\alpha)}(t_{m-})-\mathbf{x}^{(0)}(t_{m-})\right]\neq 0,\end{cases} \tag{4.48}$$

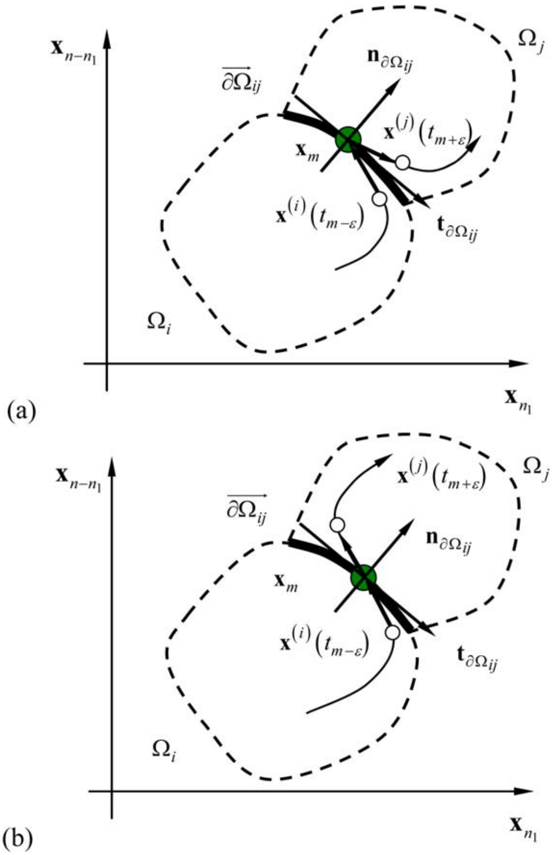
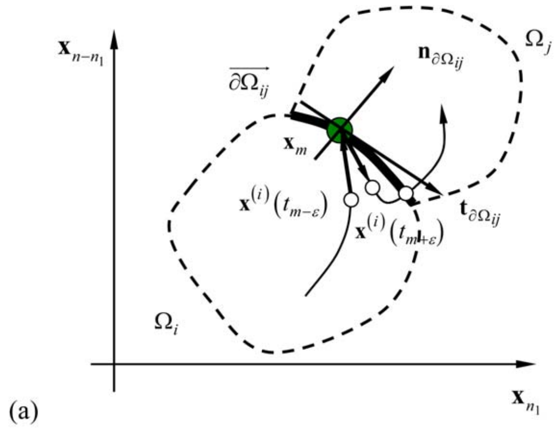


Figure 4.5. The flows passing through a boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$ for: (a) the *cusped* tangential flow ($\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(0)}_{ij} > 0$), and (b) the inflexed tangential flow $\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(0)}_{ij} < 0$.

$$\left\{\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \frac{\mathrm{d}^{k_2}}{\mathrm{d}t^{k_2}}\left[\dot{\mathbf{x}}^{(\beta)}(t_{m+}) - \mathbf{x}^{(0)}(t_{m+})\right] = 0 \quad (k_2 = 1, 2, \ldots, 2q-1), \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \frac{\mathrm{d}^{2q}}{\mathrm{d}t^{2q}}\left[\dot{\mathbf{x}}^{(\beta)}(t_{m+}) - \mathbf{x}^{(0)}(t_{m+})\right] \neq 0; \end{array}\right.$$

$$\text{either} \quad \left\{\begin{array}{l} \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] > 0 \\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m+})\right] > 0 \end{array}\right. \quad \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} > 0 \tag{4.49}$$

$$\text{or} \quad \left\{\begin{array}{l} \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] < 0 \\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m+})\right] < 0 \end{array}\right. \quad \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} < 0.$$



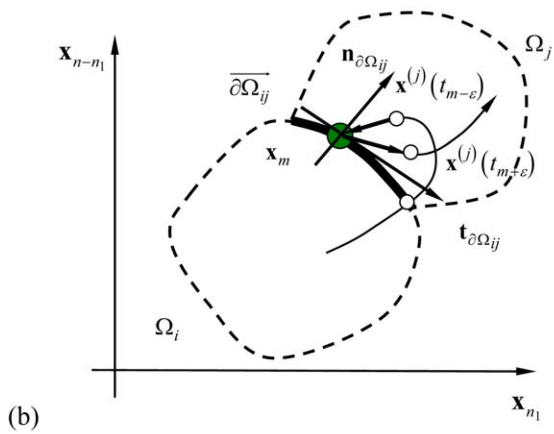


Figure 4.6. The post *cusped* tangential flows passing through a boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$ for $\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(0)}_{ij} > 0$: (a) first kind, and (b) second kind.

The theorems for such cusped and inflexed tangential flows can be developed that are similar to the ones for the transversal tangential flows of the first and second kinds. The corresponding conditions in the transverse tangential flows can be used for the tangential, input and output flows of the cusped and inflexed flows. Therefore, no further theorems are presented herein. To help understand the above definitions, the cusped and inflexed tangential flow are sketched in Figs. 4.5(a) and (b). The post cusped tangential flows of the first and second kind are presented in Figs. 4.6(a) and (b). The post inflexed tangential flows of the first and second kind are presented in Figs. 4.7(a) and (b) as well. Such tangential flows can

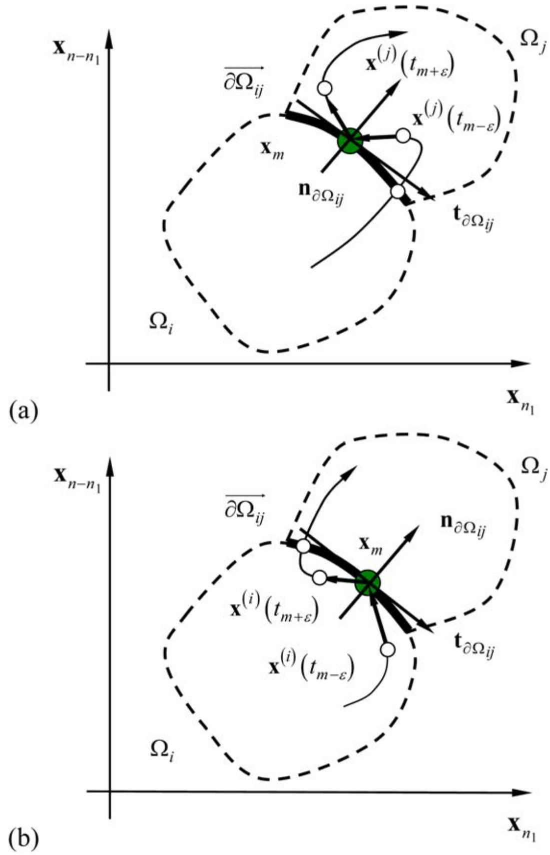


Figure 4.7. The post inflexed tangential flows passing through a boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$ for $\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(0)}_{ij} < 0$: (a) first kind, and (b) second kind.

easily be found in discontinuous dynamical systems with higher-order, nonlinear polynomial vector fields. This section will help ones investigate the singularity in nonlinear, discontinuous dynamical systems. In this book, only linear discontinuous dynamical systems are presented as sampled problems in order to verify the currently developed theory. This is because the closed form solutions for linear, discontinuous dynamical systems can be obtained easily in the accessible domains.

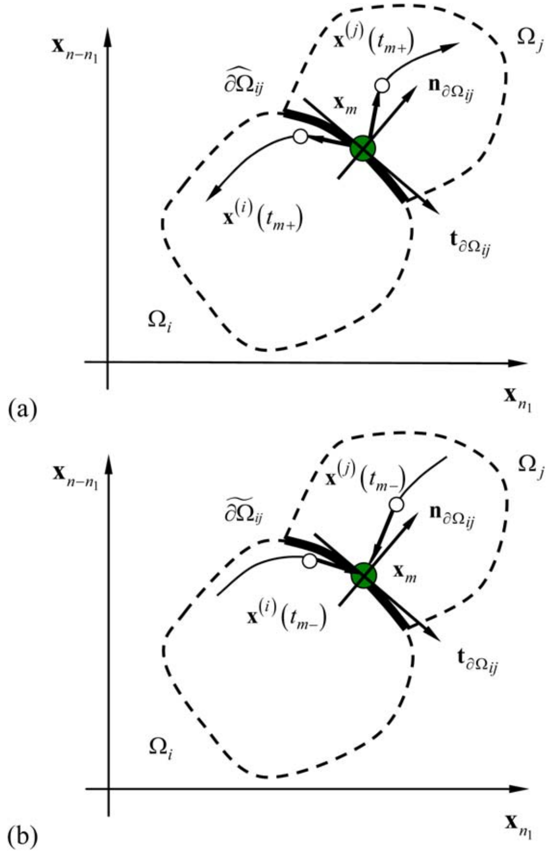


Figure 4.8. Semi-tangential nonpassable boundary set $\overline{\partial\Omega}_{ij} = \widetilde{\partial\Omega}_{ij} \cup \widehat{\partial\Omega}_{ij}$: (a) the semi-tangential sink boundary, and (b) the semi-tangential source boundary.

4.3. Nonpassable tangential flows

The properties of a flow in the semi-passable boundary are discussed. In this section, the local characteristics of flows around the nonpassable boundary will be discussed.

DEFINITION 4.9. For a discontinuous dynamical system in Eq. (2.1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_m)$. Suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}_m$ ($\alpha \in \{i, j\}$), $\mathbf{x}^{(\alpha)}(t)$ is $C^r_{[t_{m-\varepsilon}, t_m)}$-continuous ($r \geqslant 2$) for

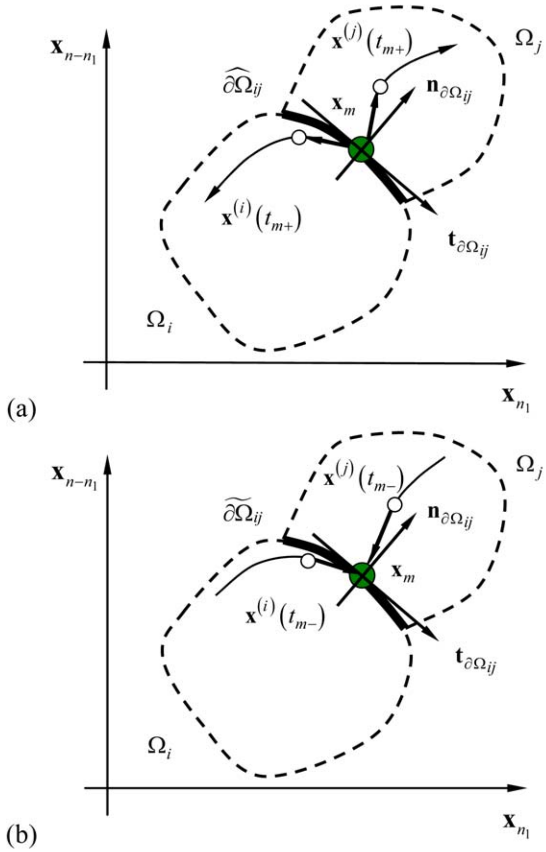


Figure 4.9. Tangential nonpassable boundary set $\overline{\partial\Omega}_{ij} = \widetilde{\partial\Omega}_{ij} \cup \widehat{\partial\Omega}_{ij}$: (a) the tangential sink boundary, and (b) the tangential source boundary.

time t and $\|\mathrm{d}^r\mathbf{x}^{(\alpha)}/\mathrm{d}t^r\| < \infty$ with

$$\begin{aligned}&\left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right]\right\} \\ &\quad\times \left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m-\varepsilon})\right]\right\} < 0. \qquad (4.50)\end{aligned}$$

(i) The flow to the nonempty boundary $\widetilde{\partial\Omega}_{ij}$ is the *semi-tangential* nonpassable flow of the first kind if $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ also satisfies

$$\text{either} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) = 0 \quad \text{or} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m-}) = 0. \qquad (4.51)$$

(ii) The flow to the nonempty boundary $\widetilde{\partial\Omega}_{ij}$ is the *tangential* nonpassable flow of the first kind, if $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ also satisfies

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) = 0 \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m-}) = 0. \tag{4.52}$$

DEFINITION 4.10. For a discontinuous dynamical system in Eq. (2.1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there is a time interval $(t_m, t_{m-\varepsilon}]$. Suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$, $\mathbf{x}^{(\alpha)}(t)$ is $C^r_{[t_{m-\varepsilon}, t_m)}$-continuous $(r \geqslant 2)$ for time t and $\|\mathrm{d}^r\mathbf{x}^{(\alpha)}/\mathrm{d}t^r\| < \infty$ with

$$\begin{aligned}&\left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(i)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right]\right\}\\ &\quad \times \left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \left[\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right]\right\} < 0.\end{aligned} \tag{4.53}$$

(i) The flow to the nonempty boundary $\widehat{\partial\Omega}_{ij}$ is the *semi-tangential* nonpassable flow of the second kind if $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ also satisfies

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+}) = 0 \quad \text{or} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+}) = 0. \tag{4.54}$$

(ii) The flow to the nonempty boundary $\widehat{\partial\Omega}_{ij}$ is the *tangential* nonpassable flow of the second kind if $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ also satisfies

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+}) = 0 \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+}) = 0. \tag{4.55}$$

The theorems for the semi-tangential and tangential nonpassable boundaries can be developed as before. The corresponding conditions in the tangential flows in the corresponding domains can be used for the tangential input or output flows on the semi-tangential and tangential nonpassable boundaries. Therefore, no further theorems are presented herein. The tangential sink and source boundaries are shown in Figs. 4.8 and 4.9 from the above definitions. The nonpassable flows occurs at the sink and source boundary bifurcations, which were also discussed in Chapter 3.

4.4. Bouncing flows

For discontinuous dynamical systems, after the semi-tangential bifurcation in Ω_j occurs at the semi-passable boundary $\overrightarrow{\partial\Omega}_{ij}$, the transversal flow may become the sliding flow on the nonpassable boundary $\widetilde{\partial\Omega}_{ij}$ or the a bouncing flow in Ω_i at both the semi-passable boundary $\overleftarrow{\partial\Omega}_{ij}$. When a flow $\mathbf{x}^{(i)}(t)$ arrives to the boundary $\overleftarrow{\partial\Omega}_{ij}$ or $\widehat{\partial\Omega}_{ij}$, the flow will bounce at such a boundary. To describe the bouncing flow, the following mathematical description is given as follows.

DEFINITION 4.11. For a discontinuous dynamical system in Eq. (2.1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals

$[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. Suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ ($\alpha \in \{i, j\}$) and $\mathbf{x}^{(\alpha)}(t)$ is $C^r_{[t_{m-\varepsilon}, t_m)}$- and $C^r_{(t_m, t_{m+\varepsilon}]}$-continuous ($r \geqslant 1$) for time t. The flow $\mathbf{x}^{(\alpha)}(t)$ in Ω_α is a bouncing flow to the boundary $\partial\Omega_{ij}$ if

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m\pm}) \neq \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta}(t_m) = 0; \tag{4.56}$$

$$\begin{aligned} &\text{either} \quad \left\{ \begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] < 0 \end{array} \right. \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta \\ &\text{or} \quad \left\{ \begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] > 0 \end{array} \right. \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha. \end{aligned} \tag{4.57}$$

DEFINITION 4.12. For a discontinuous dynamical system in Eq. (2.1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. Suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ ($\alpha \in \{i, j\}$) and $\mathbf{x}^{(\alpha)}(t)$ is $C^r_{[t_{m-\varepsilon}, t_m)}$- and $C^r_{(t_m, t_{m+\varepsilon}]}$-continuous ($r \geqslant 1$) for time t. The bouncing flow $\mathbf{x}^{(\alpha)}(t)$ in Ω_α ($\alpha \in \{i, j\}$) at $\mathbf{x}_m \in \partial\Omega_{ij}$ is of

(i) the first kind for $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ if

$$\left\{\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right]\right\} \times \left\{\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-})\right]\right\} < 0; \tag{4.58}$$

(ii) the second kind if

$$\left\{\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right]\right\} \times \left\{\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-})\right]\right\} > 0; \tag{4.59}$$

(iii) the third kind if

$$\left\{\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right]\right\} \times \left\{\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-})\right]\right\} = 0. \tag{4.60}$$

DEFINITION 4.13. For a discontinuous dynamical system in Eq. (2.1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. Suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ ($\alpha \in \{i, j\}$) and $\mathbf{x}^{(\alpha)}(t)$ is $C^r_{[t_{m-\varepsilon}, t_m)}$- and $C^r_{(t_m, t_{m+\varepsilon}]}$-continuous ($r \geqslant 1$) for time t. A bouncing flow of the third kind $\mathbf{x}^{(\alpha)}(t)$ in Ω_α ($\alpha \in \{i, j\}$) at $\mathbf{x}_m \in \partial\Omega_{ij}$ for $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ is:

(i) a normal-input bouncing flow if

$$\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] = 0; \tag{4.61}$$

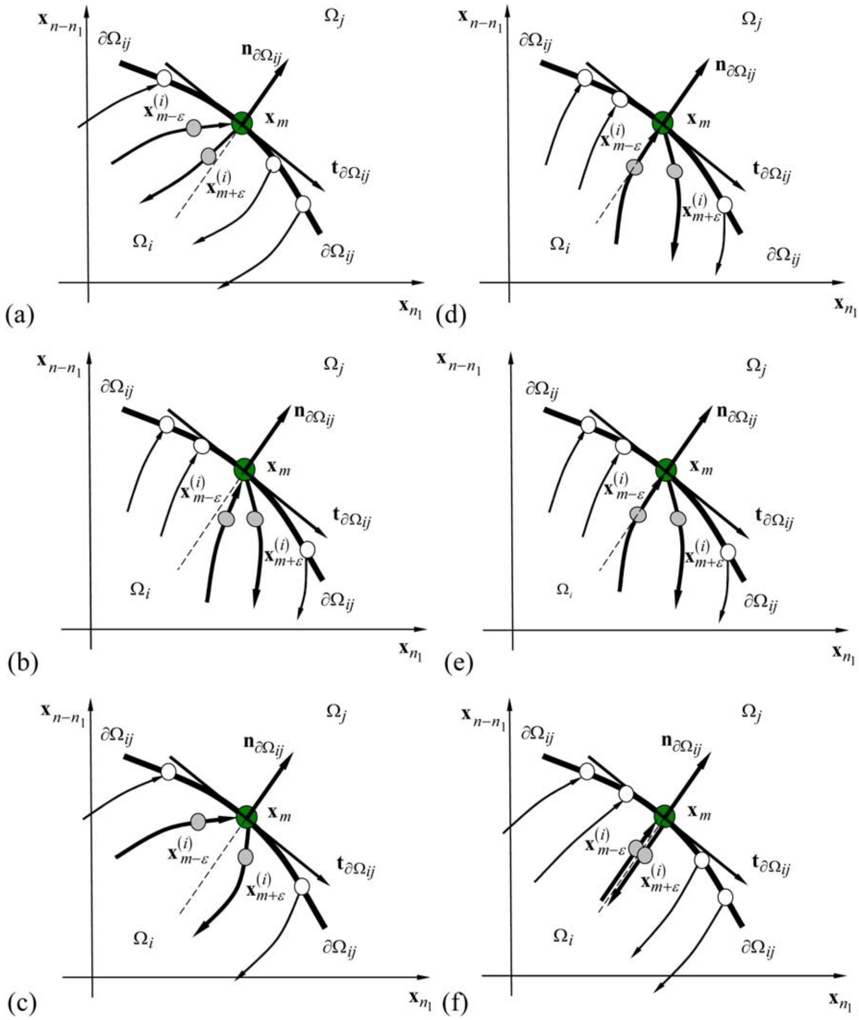


Figure 4.10. A flow in Ω_i bouncing on $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$: (a)–(c) the first- and second-kind bouncing flows, and (d)–(f) the normal-input, normal-output and complete bouncing flows. The lightly-shaded symbols represent two points ($\mathbf{x}_{m-\varepsilon}^{(i)}$ and $\mathbf{x}_{m+\varepsilon}^{(i)}$) on the flow before and after the bouncing. The bouncing point $\mathbf{x}_m$ on $\partial\Omega_{ij}$ is represented by a large circular symbol.

(ii) a normal-output bouncing flow if

$$\mathbf{t}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})\right] = 0; \tag{4.62}$$

(iii) a complete bouncing flow if

$$\begin{aligned}&\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\left[\mathbf{x}^{(\alpha)}(t_{m-})-\mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right]\\&\quad=\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon})-\mathbf{x}^{(\alpha)}(t_{m+})\right]=0.\end{aligned}\tag{4.63}$$

From the three definitions of bouncing bifurcations, the geometrical illustrations are sketched in Figs. 4.10(a)–(f). $\mathbf{n}_{\partial\Omega_{ij}}$ and $\mathbf{t}_{\partial\Omega_{ij}}$ are the normal and tangential direction of the boundary $\partial\Omega_{ij}$. The direction of $\mathbf{t}_{\partial\Omega_{ij}}\times\mathbf{n}_{\partial\Omega_{ij}}$ is the positive direction of the coordinate by the right-hand rule. The classification of bouncing bifurcations is based on the components of the flow $\mathbf{x}^{(\alpha)}(t)$ on the normal and tangential direction of the boundary $\partial\Omega_{ij}$. In Figs. 4.10(a)–(c), the first- and second-kind bouncing bifurcations are depicted. The bouncing bifurcation of the third kind is shown in Figs. 4.10(d)–(f). The lightly-shaded symbols represent two points ($\mathbf{x}^{(i)}_{m-\varepsilon}$ and $\mathbf{x}^{(i)}_{m+\varepsilon}$) on the flow before and after the bouncing. The bouncing point $\mathbf{x}_m$ on the boundary $\partial\Omega_{ij}$ is represented by a large circular symbol. This bifurcation exists only in nonsmooth dynamical systems.

THEOREM 4.9. *For a discontinuous dynamical system in Eq. (2.1),* $\mathbf{x}(t_m)\equiv\mathbf{x}_m\in\partial\Omega_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon>0$, *there are two time intervals* $[t_{m-\varepsilon},t_m)$ *and* $(t_m,t_{m+\varepsilon}]$. *Suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm})=\mathbf{x}_m$ $(\alpha\in\{i,j\})$ *and* $\mathbf{x}^{(\alpha)}(t)$ *is* $C^r_{[t_{m-\varepsilon},t_m)}$*- and* $C^r_{(t_m,t_{m+\varepsilon}]}$*-continuous* $(r\geqslant 1)$ *for time t and* $\|\mathrm{d}\mathbf{x}^r/\mathrm{d}t^r\|<\infty$. *The flow* $\mathbf{x}^{(\alpha)}(t)$ *in* Ω_α *is a bouncing flow to the boundary* $\partial\Omega_{ij}$ *iff*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\dot{\mathbf{x}}^{(\alpha)}(t_{m\pm})\neq\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\dot{\mathbf{x}}^{(\alpha)}(t_m)=0,\tag{4.64}$$

$$\begin{aligned}&\left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\left[\dot{\mathbf{x}}^{(0)}(t_{m-\varepsilon})-\dot{\mathbf{x}}^{(\alpha)}(t_{m-\varepsilon})\right]\right\}\\&\quad\times\left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\left[\dot{\mathbf{x}}^{(\alpha)}(t_{m+\varepsilon})-\dot{\mathbf{x}}^{(0)}(t_{m+\varepsilon})\right]\right\}<0.\end{aligned}\tag{4.65}$$

PROOF. This theorem can be proved following the proof procedure in Theorem 2.8. □

THEOREM 4.10. *For a discontinuous dynamical system in Eq. (2.1),* $\mathbf{x}(t_m)\equiv\mathbf{x}_m\in\partial\Omega_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon>0$, *there are two time intervals* $[t_{m-\varepsilon},t_m)$ *and* $(t_m,t_{m+\varepsilon}]$. *Suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm})=\mathbf{x}_m$ $(\alpha\in\{i,j\})$ *and* $\mathbf{F}^{(\alpha)}(t)$ *is* $C^r_{[t_{m-\varepsilon},t_m)}$*- and* $C^r_{(t_m,t_{m+\varepsilon}]}$*-continuous* $(r\geqslant 1)$ *for time t and* $\|\mathrm{d}\mathbf{x}^{r+1}/\mathrm{d}t^{r+1}\|<\infty$. *The flow* $\mathbf{x}^{(\alpha)}(t)$ *in* Ω_α *is a bouncing flow to the boundary* $\partial\Omega_{ij}$ *iff*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\mathbf{F}^{(\alpha)}(t_{m\pm})\neq\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\mathbf{F}^{(\alpha)}(t_m)=0,\tag{4.66}$$

$$\begin{aligned}&\left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\left[\mathbf{F}^{(\alpha)}(t_{m-\varepsilon})-\mathbf{F}^{(0)}(t_{m-\varepsilon})\right]\right\}\\&\quad\times\left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\left[\mathbf{F}^{(\alpha)}(t_{m+\varepsilon})-\mathbf{F}^{(0)}(t_{m+\varepsilon})\right]\right\}<0.\end{aligned}\tag{4.67}$$

PROOF. This theorem can be proved following the proof procedure in Theorem 2.9. □

THEOREM 4.11. *For a discontinuous dynamical system in Eq.* (2.1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$. *Suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$ *and* $\mathbf{x}^{(\alpha)}(t)$ *is* $C^r_{[t_{m-\varepsilon}, t_m)}$- *and* $C^r_{(t_m, t_{m+\varepsilon}]}$-*continuous* $(r \geqslant 2)$ *for time* t *and* $\|\mathrm{d}\mathbf{x}^r/\mathrm{d}t^r\| < \infty$. *The bouncing flow* $\mathbf{x}^{(\alpha)}(t)$ *in* Ω_α *is*:

- *of the first kind for* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ *iff*

$$\big[\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-\varepsilon})\big]\big[\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+\varepsilon})\big] < 0; \tag{4.68}$$

- *of the second kind iff*

$$\big[\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-\varepsilon})\big]\big[\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+\varepsilon})\big] > 0; \tag{4.69}$$

- *of the third kind iff*

$$\big[\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-\varepsilon})\big]\big[\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+\varepsilon})\big] = 0. \tag{4.70}$$

PROOF. This theorem can be proved following the proof procedure in Theorem 2.8 and using the Taylor series. □

THEOREM 4.12. *For a discontinuous dynamical system in Eq.* (2.1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$. *Suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$ *and* $\mathbf{F}^{(\alpha)}(t)$ *is* $C^r_{[t_{m-\varepsilon}, t_m)}$- *and* $C^r_{(t_m, t_{m+\varepsilon}]}$-*continuous* $(r \geqslant 1)$ *for time* t *and* $\|\mathrm{d}\mathbf{x}^{r+1}/\mathrm{d}t^{r+1}\| < \infty$. *The bouncing flow* $\mathbf{x}^{(\alpha)}(t)$ *in* Ω_α *for* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ *is*:

- *of the first kind iff*

$$\big[\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-\varepsilon})\big]\big[\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m+\varepsilon})\big] < 0; \tag{4.71}$$

- *of the second kind iff*

$$\big[\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-\varepsilon})\big]\big[\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m+\varepsilon})\big] > 0; \tag{4.72}$$

- *is of the third kind iff*

$$\big[\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-\varepsilon})\big]\big[\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m+\varepsilon})\big] = 0. \tag{4.73}$$

PROOF. This theorem can be proved using Eq. (2.1) and Theorem 4.3. □

THEOREM 4.13. *For a discontinuous dynamical system in Eq.* (2.1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals*

$[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$. *Suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$ *and* $\mathbf{x}^{(\alpha)}(t)$ *is* $C^r_{[t_{m-\varepsilon}, t_m)}$*- and* $C^r_{(t_m, t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 2)$ *for time* t *and* $\|\mathrm{d}\mathbf{x}^r/\mathrm{d}t^r\| < \infty$. *The bouncing flow of the third kind* $\mathbf{x}^{(\alpha)}(t)$ *in* Ω_α *for* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ *is*:

(i) *a normal-input bouncing flow iff*

$$\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-\varepsilon}) = 0; \tag{4.74}$$

(ii) *a normal-output bouncing flow iff*

$$\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+\varepsilon}) = 0; \tag{4.75}$$

(iii) *a complete bouncing flow iff*

$$\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-\varepsilon}) = \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+\varepsilon}) = 0. \tag{4.76}$$

PROOF. This theorem can be proved following the proof procedure in Theorem 2.8 and using the Taylor series. □

THEOREM 4.14. *For a discontinuous dynamical system in Eq.* (2.1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$. *Suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$ *and* $\mathbf{F}^{(\alpha)}(t)$ *is* $C^r_{[t_{m-\varepsilon}, t_m)}$*- and* $C^r_{(t_m, t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 1)$ *for time* t *and* $\|\mathrm{d}\mathbf{x}^{r+1}/\mathrm{d}t^{r+1}\| < \infty$. *The bouncing flow of the third kind* $\mathbf{x}^{(\alpha)}(t)$ *in* Ω_α *for* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ *is*:

(i) *a normal-input bouncing flow iff*

$$\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-\varepsilon}) = 0; \tag{4.77}$$

(ii) *a normal-output bouncing flow iff*

$$\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m+\varepsilon}) = 0; \tag{4.78}$$

(iii) *a complete bouncing flow iff*

$$\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-\varepsilon}) = \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m+\varepsilon}) = 0. \tag{4.79}$$

PROOF. The theorem can be proved using Eq. (2.1) and Theorem 4.13. □

REMARK. The theories for tangential and bouncing motions are suitable for the motion in nonsmooth dynamical systems with nonpassable boundaries.

4.5. A controlled piecewise linear system

Consider a following piecewise linear dynamical system with a control law in two domains as

$$\ddot{x} + 2d_\alpha \dot{x} + c_\alpha x = b_\alpha + Q_0 \cos \Omega t \tag{4.80}$$

where $\alpha \in \{1, 2\}$ and the two domains are partitioned by the following equation:

$$a_1 x + a_2 \dot{x} = e. \tag{4.81}$$

Once the flow leaves from a point of the boundary $\partial\Omega_{\alpha\beta}$ ($\alpha, \beta \in \{1, 2\}$, $\alpha \neq \beta$) at moment t_m into the domain Ω_α, the control force will be exerted in the corresponding domain Ω_α. The controlling force on the boundary is given through the constant force varying with the location of boundary (i.e., $b_\alpha = c_\alpha x(t_m)$, $\alpha \in \{1, 2\}$). The domains partitioned by the boundary are sketched in Fig. 4.11. The boundary is represented by the dashed line. The domains are shaded. $E = e/a_1$. The circular symbol represents an equilibrium point. Let $y = \dot{x}$, the two subdomains are expressed by

$$\begin{aligned} \Omega_1 &= \left\{(x, y) \mid a_1 x + a_2 y > e\right\} \subset \Re^2, \\ \Omega_2 &= \left\{(x, y) \mid a_1 x + a_2 y < e\right\} \subset \Re^2. \end{aligned} \tag{4.82}$$

The entire phase space is a union of the two domains,

$$\Omega = \Omega_1 \cup \Omega_2. \tag{4.83}$$

The corresponding oriented separation boundaries for $\alpha, \beta \in \{1, 2\}$ are

$$\overrightarrow{\partial\Omega}_{\alpha\beta} = \Omega_\alpha \cap \Omega_\beta = \left\{(x, y) \mid \varphi_{\alpha\beta}(x, y) \equiv a_1 x + a_2 y - e = 0\right\}. \tag{4.84}$$

From the above definitions, Eq. (4.80) gives

$$\mathbf{x} = \mathbf{F}^{(\alpha)}(\mathbf{x}, t, \boldsymbol{\mu}_\alpha, \boldsymbol{\pi}) \quad \text{for } \mathbf{x} = (x, y)^{\mathrm{T}} \in \Omega_\alpha \ (\alpha \in \{1, 2\}), \tag{4.85}$$

where

$$\mathbf{F}^{(\alpha)}(\mathbf{x}, t, \boldsymbol{\mu}_\alpha, \boldsymbol{\pi}) = (y, b_\alpha - 2d_\alpha y - c_\alpha x + Q_0 \cos \Omega t)^{\mathrm{T}}. \tag{4.86}$$

The normal and tangential vectors of the boundary $\partial\Omega_{\alpha\beta}$ are

$$\mathbf{n}_{\partial\Omega_{\alpha\beta}} = (a_1, a_2)^{\mathrm{T}} \quad \text{and} \quad \mathbf{t}_{\partial\Omega_{\alpha\beta}} = (-a_2, a_1)^{\mathrm{T}}, \tag{4.87}$$

and $\mathbf{F}^{(0)} = (-a_2, a_1)^{\mathrm{T}}$. The corresponding normal vector fields to the boundary $\partial\Omega_{\alpha\beta}$ are determined by

$$F_N^{(\alpha)} = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)} = a_1 y + a_2(b_\alpha - 2d_\alpha y - c_\alpha x + Q_0 \cos \Omega t). \tag{4.88}$$

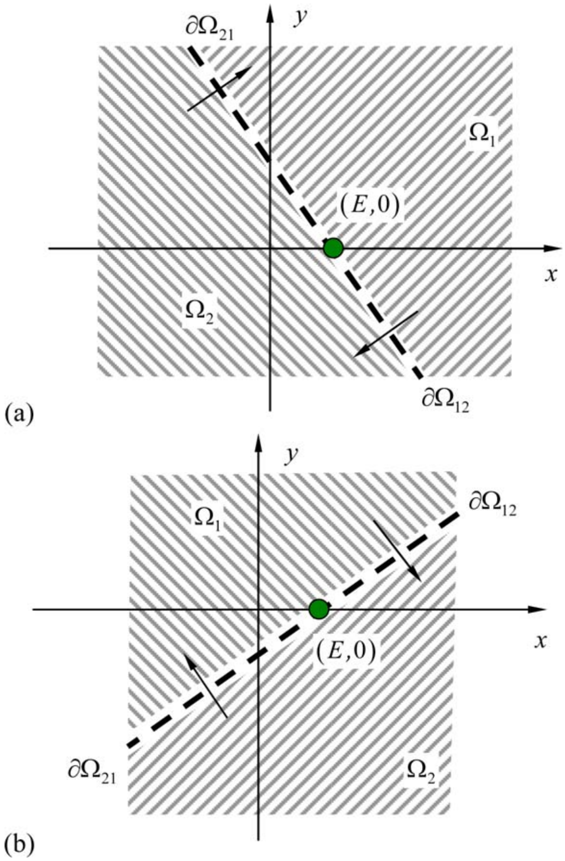


Figure 4.11. Phase plane partition for (a) $a_1a_2 > 0$, and (b) $a_1a_2 < 0$.

For each domain Ω_α ($\alpha \in \{1, 2\}$), consider the initial conditions on the boundary, the closed-form solutions are given in Appendix A. Through the closed form solutions for discontinuous dynamical systems in Eq. (4.80), the bouncing, transversal-tangential, and post-transversal-tangential flows to the separation boundary will be presented. The parameters $a_1 = a_2 = 1, c_1 = 2, c_2 = 6, d_1 = d_2 = 0.01, Q_0 = 40$ are used for illustrations of bouncing and transversal tangential flows. Select an initial condition on the boundary $(\Omega t_i, x_i, y_i) = (4.8520, -0.2150, 1.2150)$ with $\Omega = 1.6$. The phase trajectory and the corresponding normal components of vector fields for the bouncing flow in the domain Ω_2 are illustrated in Fig. 4.12. In Fig. 4.12(a), the arrow direction is the direction of the flow. The bouncing phenomenon of the flow in domain the domain Ω_2 is observed, which is labeled "Bouncing". In Fig. 4.12(b), the normal com-

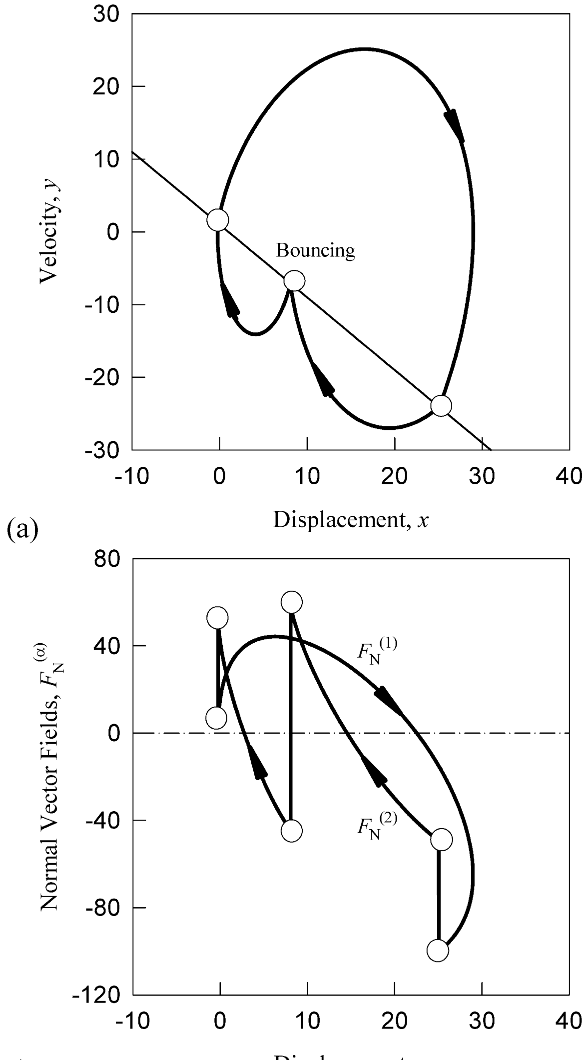


Figure 4.12. The flow bouncing on the boundary in the lower domain ($\Omega = 1.60$): (a) phase trajectory, and (b) the normal vector field to the boundary $\partial\Omega_{12}$. The initial condition is $(\Omega t_i, x_i, y_i) = (4.8520, -0.2150, 1.2150)$. ($a_1 = a_2 = 1$, $c_1 = 2$, $c_2 = 6$, $d_1 = d_2 = 0.01$, $Q_0 = 40$.)

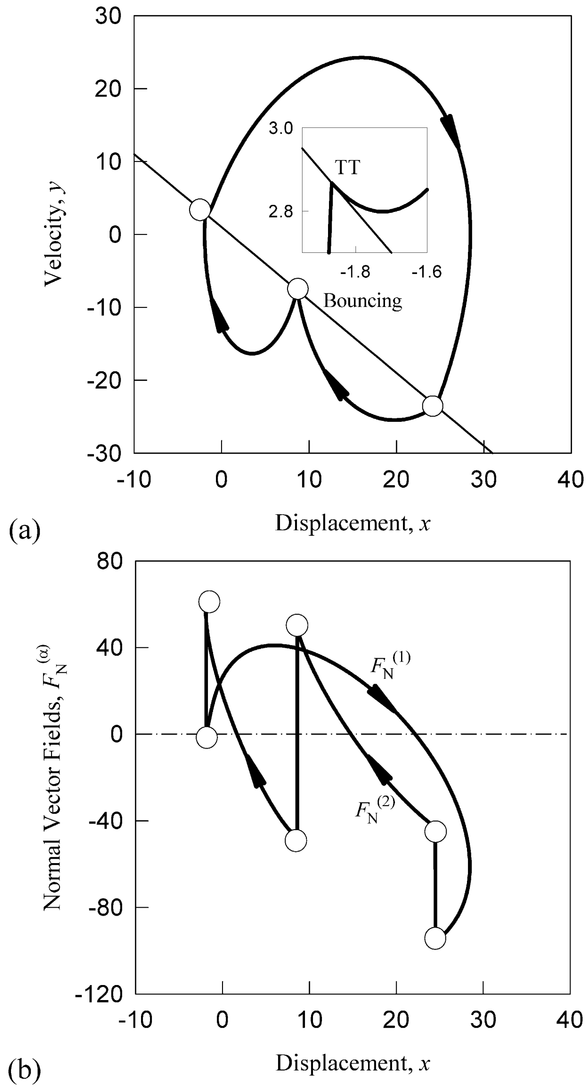


Figure 4.13. The transversal tangential flow tangential to the upper domain ($\Omega = 1.49$): (a) phase trajectory, and (b) the normal vector field to the boundary $\partial\Omega_{12}$. The initial condition is $(\Omega t_i, x_i, y_i) = (4.6458, -1.8672, 2.8672)$. ($a_1 = a_2 = 1$, $c_1 = 2$, $c_2 = 6$, $d_1 = d_2 = 0.01$, $Q_0 = 40$.) The acronym "TT" represents transversal tangential point.

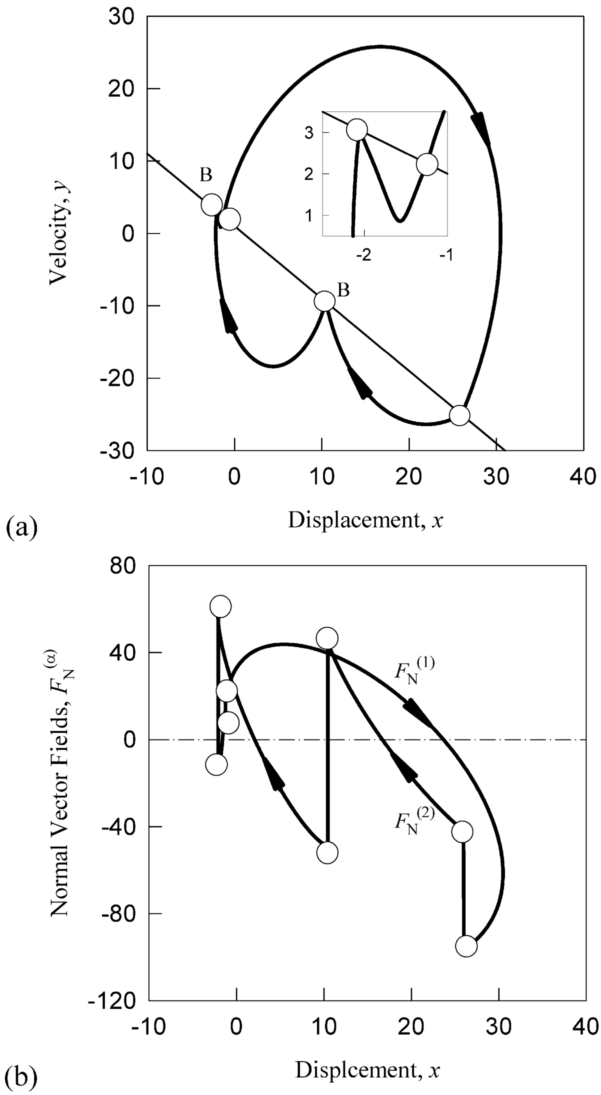


Figure 4.14. The post-transversal tangential flow ($\Omega = 1.37$): (a) phase trajectory, and (b) the normal vector field to the boundary $\partial\Omega_{12}$. The initial condition is $(\Omega t_i, x_i, y_i) = (4.3675, -2.0605, 3.0605)$. ($a_1 = a_2 = 1, c_1 = 2, c_2 = 6, d_1 = d_2 = 0.01, Q_0 = 40$.) The acronym "B" represents transversal tangential point.

ponent of vector fields to the discontinuous boundary is presented. The normal vector fields on the boundary are of the same for the flow passing over the boundary. However, for the flow bouncing on the boundary, it is observed that the normal components of the vector field before and after bouncing in the domain Ω_2 have the opposite sign. The circular symbols represent the flow switching on the boundary. In Fig. 4.13, the transversal tangential flow tangential to the upper domain with $\Omega = 1.49$ is presented. The initial condition is $(\Omega t_i, x_i, y_i) = (4.6458, -1.8672, 2.8672)$. The acronym "TT" represents transversal tangential point. Through the zoomed window, it is clearly observed that the flow tangential to the boundary just after the flow passes over the boundary. The normal component of the field for the tangential flow is zero in Fig. 4.13(b), which verify the theory developed in previous sections. For the bouncing and passability of the flow, the normal vector fields are the same as in Fig. 4.12. Finally, the post-transversal tangential flow is presented in Fig. 4.14 for $\Omega = 1.37$. The initial condition is $(\Omega t_i, x_i, y_i) = (4.3675, -2.0605, 3.0605)$. The acronym "B" represents bouncing point.

Chapter 5

Real and Imaginary Flows

The local theory for discontinuous dynamical systems on the connectable domains was presented in Chapters 2–4. The onset, existence and disappearance of the passable, nonpassable (or sink and source) flows on the boundary were investigated. In this chapter, the singularity on the separation boundary will be discussed. The singular sets of the separation boundary will be introduced. In the vicinity of such singular sets, flows hyperbolicity and parabolicity will be discussed. The real and imaginary flows in nonsmooth dynamical system will be introduced for the onset and vanishing of the sink and source flows. The imaginary flows are defined on the other domains rather than the original domain. This concept of the imaginary flows will help us understand the mechanism of sink and source flows in the vicinity of the boundary.

5.1. Singularity on boundary

In Chapter 2, the passability of flows to the separation boundary was discussed. Several separation boundaries with different passabilities will be connected together to form a complicated separation boundary. The gluing singular sets to connect all the single separation boundaries will be introduced herein. A gluing set on the boundary connects two portions of separatrix on which the flows possess two different flow directions. Therefore, this zero-dimensional gluing set has special properties, similar to the traditional singular points in dynamical systems. To make a difference between the singular points in smooth dynamical system and the singular points on the discontinuous boundary. The gluing sets are adopted herein. The definitions of passable and nonpassable boundaries are based on the flow component on the normal direction of the boundary. Therefore, the gluing sets (or surfaces) are defined as

DEFINITION 5.1. A singular set on the boundary $\partial\Omega_{ij} \subset \Re^{n-1}$

$$\Gamma_{ij} = \left\{ \mathbf{x}_k^{(0)} \in \partial\Omega_{ij} \;\middle|\; \mathbf{x} \in \Omega_\alpha \subseteq \Re^n,\ \lim_{\mathbf{x}\to\mathbf{x}_k^{(0)}} \left(\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}\right) = 0, \alpha \in \{i, j\}, k \in \mathbb{N} \right\} \subset \Re^r \tag{5.1}$$

is termed the r-dimensional, gluing singular set ($r \in \{0, 1, 2, \ldots, n-2\}$).

Notice that $\mathbb{N}$ is the natural number set. The gluing singular set is a special case of the corner point sets. This gluing surface can be either static or dynamic. The static gluing sets can be determined from the equilibrium points for equations of the sliding dynamics (i.e., $\lim_{\mathbf{x}\to\mathbf{x}_k^{(0)}}(\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\dot{\mathbf{x}}) = 0$). If the equation of motion for sliding dynamics in friction-induced vibration did not have equilibrium, the dynamical gluing sets also exist (i.e., $\lim_{\mathbf{x}\to\mathbf{x}_k^{(0)}}(\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\dot{\mathbf{x}}) \neq 0$). For 2-D discontinuous systems, the gluing singular surfaces will shrink into points. For 3-D discontinuous, the gluing singular surfaces will be lines or points.

DEFINITION 5.2. A singular set on the boundary $\partial\Omega_{ij} \subset \Re^{n-1}$

$$\Gamma_{ij}^{(\alpha)} = \Big\{\mathbf{x}_k^{(0)} \in \partial\Omega_{ij} \;\Big|\; \mathbf{x} \in \Omega_\alpha \subseteq \Re^n,\ \lim_{\mathbf{x}\to\mathbf{x}_k^{(0)}}\big(\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\dot{\mathbf{x}}\big) = 0, \alpha = \{i \text{ or } j\},\\ k \in \mathbb{N}\Big\} \subseteq \Re^r \tag{5.2}$$

is termed the r-dimensional, input or output, semi-gluing, singular set on the boundary ($r \in \{0, 1, 2, \ldots, n-2\}$).

The above definition $\Gamma_{ij}^{(\alpha)}$ indicates the switching of the flow direction at the singular sets on the side of Ω_α

DEFINITION 5.3. A countable set on the boundary $\partial\Omega_{ij} \subset \Re^{n-1}$

$$\Gamma_{ij}^{(0)} = \Big\{\mathbf{x}_k^{(0)} \in \partial\Omega_{ij} \;\Big|\; \mathbf{x} \in \Omega_\alpha \subseteq \Re^n,\ \lim_{\mathbf{x}\to\mathbf{x}_k^{(0)}}\big(\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\dot{\mathbf{x}}\big) = 0, \alpha = \{i \text{ and } j\},\\ k \in \mathbb{N}\Big\} \tag{5.3}$$

is termed the r-dimensional, full-gluing, singular point set ($r \in \{0, 1, 2, \ldots, n-2\}$).

The foregoing definition $\Gamma_{ij}^{(0)}$ indicates the switching of the flow direction at the singular point on both sides of Ω_α. The singular, gluing set is $\Gamma_{ij} = \Gamma_{ij}^{(i)} \cup \Gamma_{ij}^{(j)} \cup \Gamma_{ij}^{(0)}$.

5.2. Hyperbolicity and parabolicity

To investigate the dynamical behaviors in the neighborhood of the r-dimensional, gluing singular sets, the δ-subdomains and boundaries relative to the gluing sets are defined as

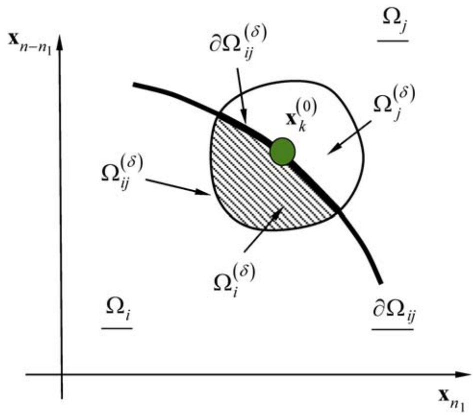


Figure 5.1. The δ-subdomains and sub-boundary in the neighborhood of the gluing sets $\mathbf{x}_k^{(0)} \in \Gamma_{ij}$.

DEFINITION 5.4. The δ-subdomains and boundaries are

$$\Omega_\alpha^{(\delta)} = \big\{\mathbf{x} \in \Omega_\alpha \;\big|\; \text{for a given } \delta > 0,\ \big\|\mathbf{x} - \mathbf{x}_k^{(0)}\big\| < \delta,\ \mathbf{x}_k^{(0)} \in \Gamma_{ij}, \alpha \in \{i, j\} \text{ for } k \in \mathbb{N}\big\}, \tag{5.4}$$

$$\partial\Omega_{ij}^{(\delta)} = \big\{\mathbf{x}_m \in \partial\Omega_{ij} \;\big|\; \text{for a given } \delta > 0,\ \big\|\mathbf{x}_m - \mathbf{x}_k^{(0)}\big\| < \delta,\ \mathbf{x}_k^{(0)} \in \Gamma_{ij}, \alpha \in \{i, j\} \text{ for } k \in \mathbb{N}\big\}, \tag{5.5}$$

$$\Omega_{ij}^{(\delta)} = \Omega_i^{(\delta)} \cup \Omega_j^{(\delta)} \cup \partial\Omega_{ij}^{(\delta)}. \tag{5.6}$$

From the above definition, the δ-subdomains and boundary in the neighborhood of $\mathbf{x}_k^{(0)} \in \Gamma_{ij}$ are presented in Fig. 5.1. The δ-boundary $\partial\Omega_{ij}^{(\delta)}$ is represented by the dark curve. The gluing set is expressed by the circular symbol. The δ-subdomains $\Omega_i^{(\delta)}$ and $\Omega_j^{(\delta)}$ are expressed by the shaded and white areas, respectively. Through the flow possibility, a 3-D sketch for the δ-subdomains and singular sets with several separation surfaces are arranged in Fig. 5.2.

DEFINITION 5.5. For a discontinuous dynamical system in Eq. (2.1), there is a gluing point $\mathbf{x}_k^{(0)} \in \Gamma_{ij}$ on $\partial\Omega_{ij}$. For an arbitrarily small $\delta > 0$, there are two points $\mathbf{x}_{m_1}, \mathbf{x}_{m_2} \in \partial\Omega_{ij}^{(\delta)}$ and $\mathbf{x}^{(\alpha)}(t_{m_1+}), \mathbf{x}^{(\alpha)}(t_{m_2-}) \in \Omega_\alpha^{(\delta)}$ ($\alpha \in \{i, j\}$). Suppose $\mathbf{x}^{(\alpha)}(t_{m_1+}) = \mathbf{x}_{m_1}$ and $\mathbf{x}^{(\alpha)}(t_{m_2-}) = \mathbf{x}_{m_2}$, $\mathbf{x}^{(\alpha)}(t)$ is $C^r_{(t_{m+}, t_{n-})}$-continuous ($r \geqslant 2$) and $\|\mathrm{d}^r\mathbf{x}^{(\alpha)}/\mathrm{d}t^r\| < \infty$ in Ω_α. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $(t_{m_1+}, t_{m_1+\varepsilon}]$ and $[t_{m_2-\varepsilon}, t_{m_2-})$. For $\mathbf{x}^{(\alpha)}(t_{m_1+\varepsilon}), \mathbf{x}^{(\alpha)}(t_{m_2-\varepsilon}) \in \Omega_\alpha^{(\delta)}$ and

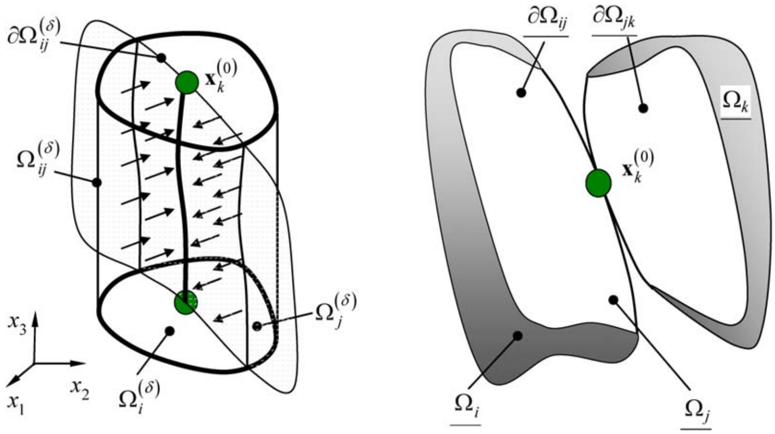


Figure 5.2. A 3-D illustration of the δ-subdomains and sub-boundary and singular points with several separation surfaces.

$\alpha \neq \beta$, the following conditions hold:

$$\begin{aligned}
&\text{either} \quad \left\{\begin{array}{l} \left(\mathbf{n}_{\partial\Omega_{ij}}^{m_1}\right)^{\mathrm{T}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m_1+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m_1+})\right] > 0 \\ \left(\mathbf{n}_{\partial\Omega_{ij}}^{m_2}\right)^{\mathrm{T}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m_2-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m_2-})\right] < 0 \end{array}\right. \\
&\qquad\qquad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha \\
&\text{or} \quad \left\{\begin{array}{l} \left(\mathbf{n}_{\partial\Omega_{ij}}^{m_1}\right)^{\mathrm{T}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m_1+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m_1+})\right] < 0 \\ \left(\mathbf{n}_{\partial\Omega_{ij}}^{m_2}\right)^{\mathrm{T}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m_2-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m_2-})\right] > 0 \end{array}\right. \\
&\qquad\qquad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta .
\end{aligned} \tag{5.7}$$

The gluing singular point $\mathbf{x}_k^{(0)} \in \Gamma_{ij}$ is parabolic on the side of Ω_α if

$$\begin{aligned}
&\text{either} \quad \left\{\begin{array}{l} \left(\mathbf{t}_{\partial\Omega_{ij}}^{m_1}\right)^{\mathrm{T}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m_1+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m_1+})\right] > 0 \\ \left(\mathbf{t}_{\partial\Omega_{ij}}^{m_2}\right)^{\mathrm{T}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m_2-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m_2-})\right] > 0 \end{array}\right. \\
&\qquad\qquad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha \\
&\text{or} \quad \left\{\begin{array}{l} \left(\mathbf{t}_{\partial\Omega_{ij}}^{m_1}\right)^{\mathrm{T}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m_1+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m_1+})\right] < 0 \\ \left(\mathbf{t}_{\partial\Omega_{ij}}^{m_2}\right)^{\mathrm{T}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m_2-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m_2-})\right] < 0 \end{array}\right. \\
&\qquad\qquad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta .
\end{aligned} \tag{5.8}$$

The gluing singular point $\mathbf{x}_k^{(0)} \in \Gamma_{ij}$ is hyperbolic on the side of Ω_α if

$$\begin{aligned}
&\text{either} \quad \begin{cases} \left(\mathbf{t}_{\partial\Omega_{ij}}^{m_1}\right)^{\mathrm{T}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m_1+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m_1+})\right] < 0 \\ \left(\mathbf{t}_{\partial\Omega_{ij}}^{m_2}\right)^{\mathrm{T}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m_2-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m_2-})\right] < 0 \end{cases} \\
&\qquad\quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha \\
&\text{or} \quad \begin{cases} \left(\mathbf{t}_{\partial\Omega_{ij}}^{m_1}\right)^{\mathrm{T}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m_1+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m_1+})\right] > 0 \\ \left(\mathbf{t}_{\partial\Omega_{ij}}^{m_2}\right)^{\mathrm{T}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m_2-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m_2-})\right] > 0 \end{cases} \\
&\qquad\quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta.
\end{aligned} \tag{5.9}$$

Note that $\mathbf{t}_{\partial\Omega_{ij}}^{m_1}$ and $\mathbf{n}_{\partial\Omega_{ij}}^{m_1}$ are the tangential and normal vectors relative to the point $\mathbf{x}_{m_1} \in \partial\Omega_{ij}$ but $\mathbf{x}_{m_1} \notin \Gamma_{ij}$. $\mathbf{t}_{\partial\Omega_{ij}}^{m_1} \cdot \mathbf{n}_{\partial\Omega_{ij}}^{m_1} > 0$. From the above definition, the hyperbolicity and parabolicity of the gluing point $\mathbf{x}_k^{(0)} \in \Gamma_{ij}$ are illustrated respectively in Figs. 5.3 and 5.4 as $\delta \to 0$ and $\varepsilon \to 0$. The domain in the dashed boundary is $\Omega_{ij}^{(\delta)}$. The flow $\mathbf{x}^{(\alpha)}(t)$ in $\Omega_\alpha^{(\delta)}$ for $t \in (t_{m_1+}, t_{m_2-})$ will not have any other sets intersected with the boundary $\partial\Omega_{ij}$. The hyperbolicity and parabolicity of the gluing singular sets on both sides of the boundary are different from the ones in continuous dynamical systems. Such properties generate the motion complexity in nonsmooth dynamical systems. If all $\mathbf{x}_k^{(0)} \in \Gamma_{ij}$ are parabolic, the singular gluing surface Γ_{ij} is parabolic. However, if all $\mathbf{x}_k^{(0)} \in \Gamma_{ij}$ are hyperbolic, the singular gluing surface Γ_{ij} is hyperbolic.

THEOREM 5.1. *For a discontinuous dynamical system in Eq.* (2.1), *there is a gluing point* $\mathbf{x}_k^{(0)} \in \Gamma_{ij}$ *on* $\partial\Omega_{ij}$. *For an arbitrarily small* $\delta > 0$, *there are two points* $\mathbf{x}_{m_1}, \mathbf{x}_{m_2} \in \partial\Omega_{ij}^{(\delta)}$ *and* $\mathbf{x}^{(\alpha)}(t_{m_1+}), \mathbf{x}^{(\alpha)}(t_{m_2-}) \in \Omega_\alpha^{(\delta)}$ $(\alpha \in \{i, j\})$. *Suppose* $\mathbf{x}^{(\alpha)}(t_{m_1+}) = \mathbf{x}_{m_1}$ *and* $\mathbf{x}^{(\alpha)}(t_{m_2-}) = \mathbf{x}_{m_2}$, $\mathbf{x}^{(\alpha)}(t)$ *is* $C^r_{(t_{m+}, t_{n-})}$*-continuous* $(r \geqslant 2)$, $\|\mathrm{d}^r\mathbf{x}^{(\alpha)}/\mathrm{d}t^r\| < \infty$ *in* Ω_α *and* $\alpha \neq \beta$. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals* $(t_{m_1+}, t_{m_1+\varepsilon}]$ *and* $[t_{m_2-\varepsilon}, t_{m_2-})$, *and the following relation holds*:

$$\begin{aligned}
&\textit{either} \quad \left(\mathbf{n}_{\partial\Omega_{ij}}^{m_1}\right)^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m_1+}) > 0 \quad \textit{and} \quad \left(\mathbf{n}_{\partial\Omega_{ij}}^{m_2}\right)^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m_2-}) < 0 \\
&\qquad\quad \textit{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha \\
&\textit{or} \quad \left(\mathbf{n}_{\partial\Omega_{ij}}^{m_1}\right)^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m_1+}) < 0 \quad \textit{and} \quad \left(\mathbf{n}_{\partial\Omega_{ij}}^{m_2}\right)^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m_2-}) > 0 \\
&\qquad\quad \textit{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta.
\end{aligned} \tag{5.10}$$

The gluing singular point $\mathbf{x}_k^{(0)} \in \Gamma_{ij}$ *is parabolic on the side of* Ω_α *iff*

$$\textit{either} \quad \left(\mathbf{n}_{\partial\Omega_{ij}}^{m_1}\right)^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m_1+}) > 0 \quad \textit{and} \quad \left(\mathbf{n}_{\partial\Omega_{ij}}^{m_2}\right)^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m_2-}) > 0$$

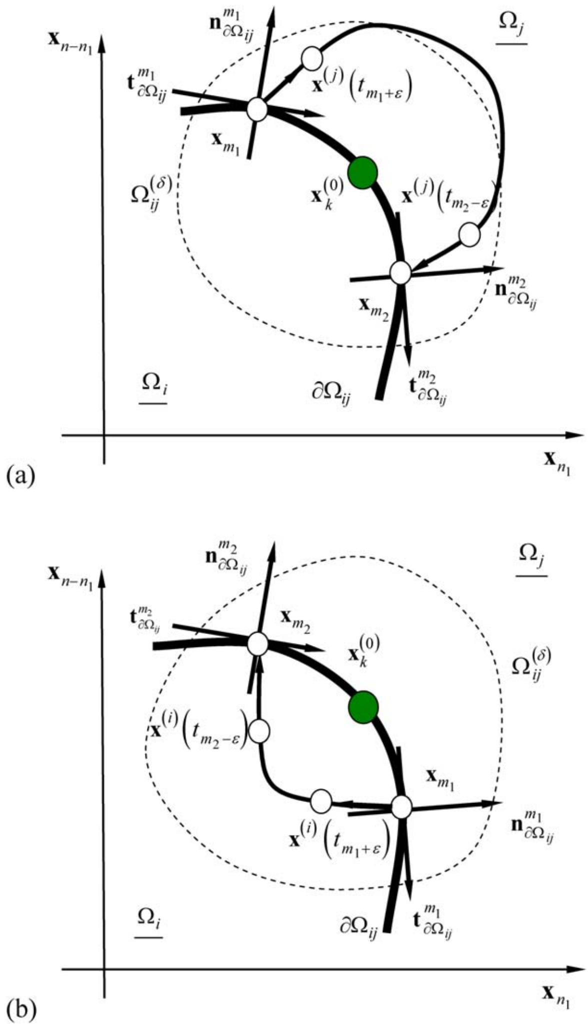


Figure 5.3. Parabolic flows in the δ-domain $\Omega^{(\delta)}_{ij}$ of $\mathbf{x}^{(0)}_k \in \Gamma_{ij}$ on the side of (a) Ω_j and (b) Ω_i. The boundary $\partial\Omega_{ij}$ possesses a normal vector $\mathbf{n}_{\partial\Omega_{ij}}$ pointing to the domain Ω_j. The domain in the dashed boundary is $\Omega^{(\delta)}_{ij}$.

$$
\begin{aligned}
&\quad\quad \textit{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha \\
\textit{or} \quad &\left(\mathbf{n}^{m_1}_{\partial\Omega_{ij}}\right)^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m_1+}) < 0 \quad \textit{and} \quad \left(\mathbf{n}^{m_2}_{\partial\Omega_{ij}}\right)^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m_2-}) < 0 \\
&\quad\quad \textit{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta .
\end{aligned}
\tag{5.11}
$$

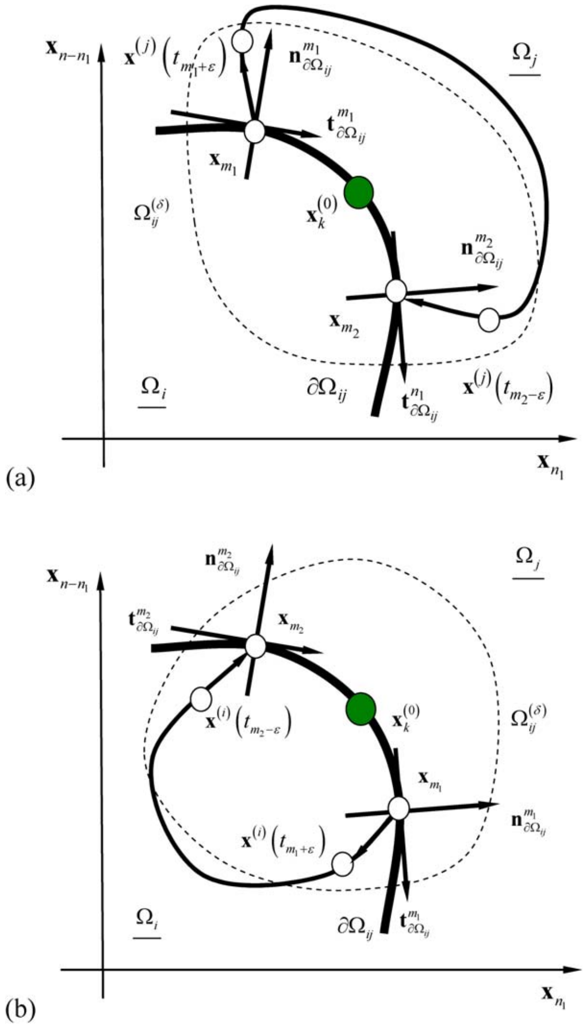


Figure 5.4. Hyperbolic flows in the δ-domain $\Omega^{(\delta)}_{ij}$ of $\mathbf{x}^{(0)}_k \in \Gamma_{ij}$ on the side of (a) Ω_j and (b) Ω_i. The boundary $\partial\Omega_{ij}$ possesses a normal vector $\mathbf{n}_{\partial\Omega_{ij}}$ pointing to the domain Ω_j. The domain in the dashed boundary is $\Omega^{(\delta)}_{ij}$.

The gluing singular point $\mathbf{x}^{(0)}_k \in \Gamma_{ij}$ *is hyperbolic on the side of* Ω_α *iff*

$$\text{either}\quad \left(\mathbf{n}^{m_1}_{\partial\Omega_{ij}}\right)^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m_1+}) < 0 \quad \text{and} \quad \left(\mathbf{n}^{m_2}_{\partial\Omega_{ij}}\right)^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m_2-}) < 0$$
$$\text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha \tag{5.12}$$

$$\begin{array}{ll} or & \big(\mathbf{n}_{\partial\Omega_{ij}}^{m_1}\big)^{\mathrm{T}}\cdot\dot{\mathbf{x}}^{(\alpha)}(t_{m_1+})>0 \quad and \quad \big(\mathbf{n}_{\partial\Omega_{ij}}^{m_2}\big)^{\mathrm{T}}\cdot\dot{\mathbf{x}}^{(\alpha)}(t_{m_2-})>0 \\ & for\ \mathbf{n}_{\partial\Omega_{\alpha\beta}}\to\Omega_\beta. \end{array}$$

PROOF. Following the proof procedure of Theorem 2.1, the theorem can be proved. □

THEOREM 5.2. *For a discontinuous dynamical system in Eq. (2.1), there is a gluing point* $\mathbf{x}_k^{(0)}\in\Gamma_{ij}$ *on* $\partial\Omega_{ij}$. *For an arbitrarily small* $\delta>0$, *there are two points* $\mathbf{x}_m,\mathbf{x}_n\in\partial\Omega_{ij}^{(\delta)}$ *and* $\mathbf{x}^{(\alpha)}(t_{m_1+}),\mathbf{x}^{(\alpha)}(t_{m_1-})\in\Omega_\alpha^{(\delta)}$ $(\alpha\in\{i,j\})$. *Suppose* $\mathbf{x}^{(\alpha)}(t_{m_1+})=\mathbf{x}_{m_1}$ *and* $\mathbf{x}^{(\alpha)}(t_{m_2-})=\mathbf{x}_{m_2}$, $\mathbf{F}^{(\alpha)}(t)$ *is* $C^r_{(t_{m_1+},t_{m_2-})}$*-continuous* $(r\geqslant 1)$, $\|\mathrm{d}^{r+1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r+1}\|<\infty$ *in* Ω_α, $\alpha\neq\beta$ *and the following relation holds*:

$$\begin{array}{ll} either & \big(\mathbf{n}_{\partial\Omega_{ij}}^{m_1}\big)^{\mathrm{T}}\cdot\mathbf{F}^{(\alpha)}(t_{m_1+})>0 \quad and \quad \big(\mathbf{n}_{\partial\Omega_{ij}}^{m_2}\big)^{\mathrm{T}}\cdot\mathbf{F}^{(\alpha)}(t_{m_2-})<0 \\ & for\ \mathbf{n}_{\partial\Omega_{\alpha\beta}}\to\Omega_\alpha \\ or & \big(\mathbf{n}_{\partial\Omega_{ij}}^{m_1}\big)^{\mathrm{T}}\cdot\mathbf{F}^{(\alpha)}(t_{m_1+})<0 \quad and \quad \big(\mathbf{n}_{\partial\Omega_{ij}}^{m_2}\big)^{\mathrm{T}}\cdot\mathbf{F}^{(\alpha)}(t_{m_2-})>0 \\ & for\ \mathbf{n}_{\partial\Omega_{\alpha\beta}}\to\Omega_\beta. \end{array} \tag{5.13}$$

The gluing point $\mathbf{x}_k^{(0)}\in\Gamma_{ij}$ *is parabolic on the side of* Ω_α *iff*

$$\begin{array}{ll} either & \big(\mathbf{n}_{\partial\Omega_{ij}}^{m_1}\big)^{\mathrm{T}}\cdot\mathbf{F}^{(\alpha)}(t_{m_1+})>0 \quad and \quad \big(\mathbf{n}_{\partial\Omega_{ij}}^{m_2}\big)^{\mathrm{T}}\cdot\mathbf{F}^{(\alpha)}(t_{m_2-})>0 \\ & for\ \mathbf{n}_{\partial\Omega_{\alpha\beta}}\to\Omega_\alpha \\ or & \big(\mathbf{n}_{\partial\Omega_{ij}}^{m_1}\big)^{\mathrm{T}}\cdot\mathbf{F}^{(\alpha)}(t_{m_1+})<0 \quad and \quad \big(\mathbf{n}_{\partial\Omega_{ij}}^{m_2}\big)^{\mathrm{T}}\cdot\mathbf{F}^{(\alpha)}(t_{m_2-})<0 \\ & for\ \mathbf{n}_{\partial\Omega_{\alpha\beta}}\to\Omega_\beta. \end{array} \tag{5.14}$$

The gluing point $\mathbf{x}_k^{(0)}\in\Gamma_{ij}$ *is hyperbolic on the side of* Ω_α *iff*

$$\begin{array}{ll} either & \big(\mathbf{n}_{\partial\Omega_{ij}}^{m_1}\big)^{\mathrm{T}}\cdot\mathbf{F}^{(\alpha)}(t_{m_1+})<0 \quad and \quad \big(\mathbf{n}_{\partial\Omega_{ij}}^{m_2}\big)^{\mathrm{T}}\cdot\mathbf{F}^{(\alpha)}(t_{m_2-})<0 \\ & for\ \mathbf{n}_{\partial\Omega_{\alpha\beta}}\to\Omega_\alpha \\ or & \big(\mathbf{n}_{\partial\Omega_{ij}}^{m_1}\big)^{\mathrm{T}}\cdot\mathbf{F}^{(\alpha)}(t_{m_1+})>0 \quad and \quad \big(\mathbf{n}_{\partial\Omega_{ij}}^{m_2}\big)^{\mathrm{T}}\cdot\mathbf{F}^{(\alpha)}(t_{m_2-})>0 \\ & for\ \mathbf{n}_{\partial\Omega_{\alpha\beta}}\to\Omega_\beta. \end{array} \tag{5.15}$$

PROOF. Following the proof procedure of Theorem 2.2, the theorem can be proved. □

If the gluing singular point $\mathbf{x}_k^{(0)}\in\Gamma_{ij}^{(0)}$ on both sides of boundary possesses the hyperbolicity in the corresponding δ-domain, the hyperbolic motion will appear. If the gluing singular point $\mathbf{x}_k^{(0)}\in\Gamma_{ij}^{(0)}$ on both sides of boundary experiences parabolicity in the corresponding δ-domain, the parabolic motion will be observed.

However, due to the discontinuity, the parabolicity and hyperbolicity of the gluing singular point $\mathbf{x}_k^{(0)} \in \Gamma_{ij}^{(0)}$ on both sides of the boundary cannot occur at the same time always. Therefore, the C-motion will appear. The mathematical definition is given as follows.

DEFINITION 5.6. In $\Omega_{ij}^{(\delta)}$ for $\mathbf{x}_k^{(0)} \in \Gamma_{ij}$, there is $\mathbf{x}^{(\alpha)}(t)$ in $\Omega_{\alpha}^{(\delta)}$ $(\alpha \in \{i, j\})$ and $\mathbf{x}^{(\beta)}(t)$ in $\Omega_{\beta}^{(\delta)}$ $(\beta \in \{i, j\}, \alpha \neq \beta)$. Three possible motions exist:

(i) The flow in $\Omega_{ij}^{(\delta)}$ is termed a *C-flow* around the gluing point $\mathbf{x}_k^{(0)}$ if $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ possess the hyperbolicity and parabolicity to $\mathbf{x}_k^{(0)}$, respectively.
(ii) The flow in $\Omega_{ij}^{(\delta)}$ is termed a *hyperbolic flow* around the gluing point $\mathbf{x}_k^{(0)}$ if $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ possess the hyperbolicity to $\mathbf{x}_k^{(0)}$.
(iii) The flow in $\Omega_{ij}^{(\delta)}$ is termed a *parabolic flow* around the gluing point $\mathbf{x}_k^{(0)}$ if $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ possess the parabolicity to $\mathbf{x}_k^{(0)}$.

5.3. Boundary formation

In the nonsmooth dynamic system, the separation boundary often consists of several semi-passable boundaries, nonpassable boundaries and gluing singular points. Consider two semi-passable boundary sets $\overrightarrow{\partial\Omega}_{ij}$ and $\overleftarrow{\partial\Omega}_{ij}$ with a gluing point, i.e.,

$$\overleftrightarrow{\partial\Omega}_{ij} = \overrightarrow{\partial\Omega}_{ij} \cup \overleftarrow{\partial\Omega}_{ij} \cup \Gamma_{ij}^{(0)}. \tag{5.16}$$

Based on the above boundary, the hyperbolic, parabolic and C-shape motions in the δ-domain of the gluing point $\mathbf{x}_k^{(0)}$ can exist. The corresponding phase portraits in the δ-domain of the gluing point are sketched in Figs. 5.5 and 5.6, respectively. The largest, solid circular circle is the singular gluing point $\mathbf{x}_k^{(0)} \in \Gamma_{ij}$. The boldest solid curve with circular symbols is the discontinuous boundary set. On the semi-passable boundary $\overrightarrow{\partial\Omega}_{ij}$ (or $\overleftarrow{\partial\Omega}_{ij}$), the flow depicted by the smaller solid curves passes through the boundary from the domain Ω_i into Ω_j (or Ω_j into Ω_i). Generally, an open chain boundary consisting of $\overrightarrow{\partial\Omega}_{ij}^{(n_1)}$ and $\overleftarrow{\partial\Omega}_{ij}^{(n_2)}$ with $\Gamma_{ij}^{(0)(n_3)}$ possesses the following structures as

$$\overleftrightarrow{\partial\Omega}_{ij} = \bigcup_{n_1=1}^{k_1} \overrightarrow{\partial\Omega}_{ij}^{(n_1)} \cup \bigcup_{n_2=1}^{k_2} \overleftarrow{\partial\Omega}_{ij}^{(n_2)} \cup \bigcup_{n_3=1}^{k_1+k_2-1} \Gamma_{ij}^{(0)(n_3)} \tag{5.17}$$

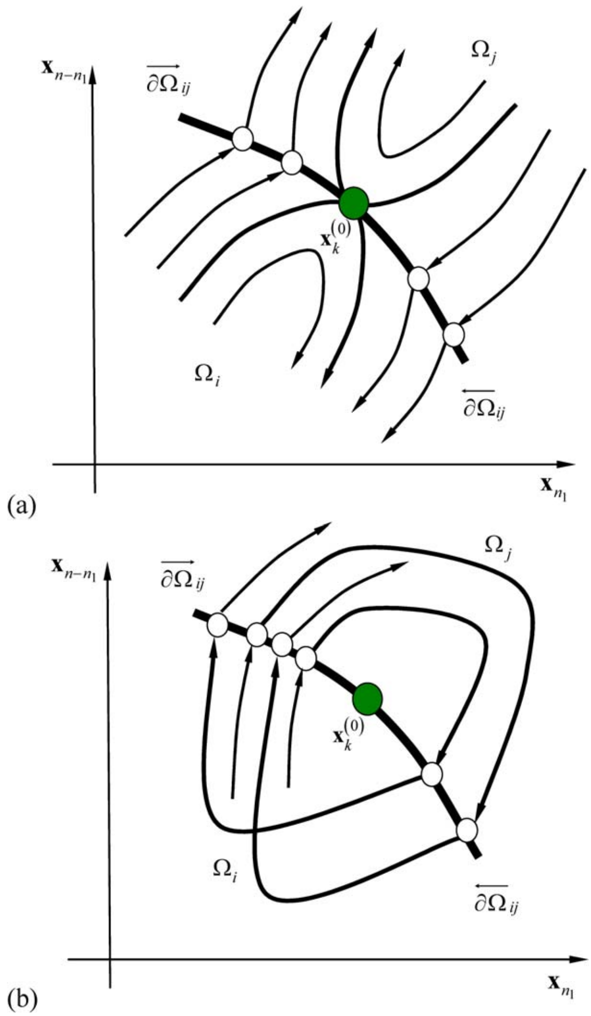


Figure 5.5. Phase portraits for (a) hyperbolic, (b) parabolic flows in the δ-domain near the discontinuous boundary set $\partial\Omega_{ij} = \overrightarrow{\partial\Omega}_{ij} \cup \overleftarrow{\partial\Omega}_{ij} \cup \Gamma_{ij}^{(0)}$. The largest, solid circular circles denote the gluing sets $\Gamma_{ij}^{(i)}$ and $\Gamma_{ij}^{(j)}$. The boldest solid curve with circular symbols is the entire discontinuous boundary set.

where two integers satisfy $|k_1 - k_2| \leqslant 1$. A closed passable boundary is formed as

$$\overleftrightarrow{\partial\Omega}_{ij} = \bigcup_{n_1=1}^{k} \overrightarrow{\partial\Omega}_{ij}^{(n_1)} \cup \bigcup_{n_2=1}^{k} \overleftarrow{\partial\Omega}_{ij}^{(n_2)} \cup \bigcup_{n_3=1}^{2n} \Gamma_{ij}^{0(n_3)}. \tag{5.18}$$

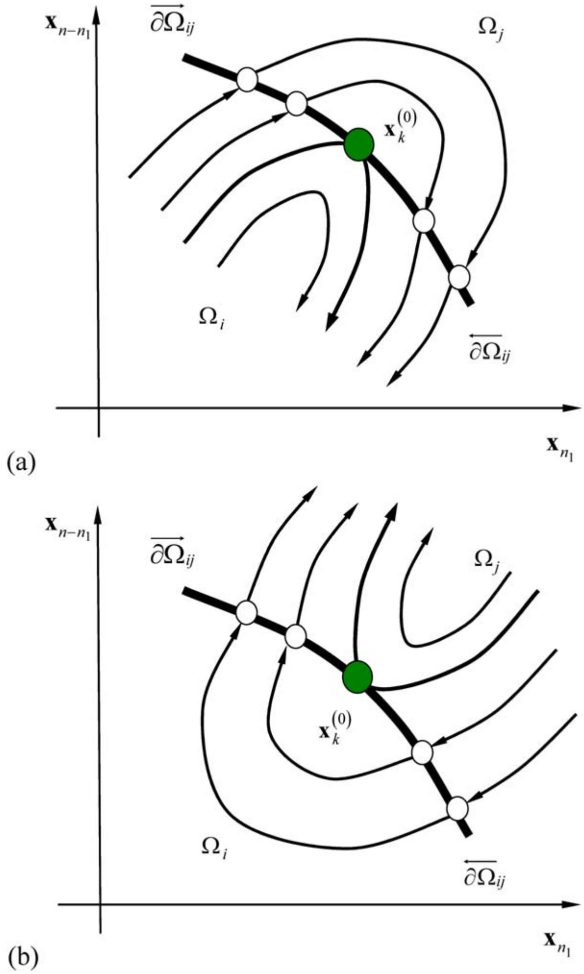


Figure 5.6. Phase portraits for two C-motions in the δ-domain near the discontinuous boundary set $\partial\Omega_{ij} = \overrightarrow{\partial\Omega}_{ij} \cup \overleftarrow{\partial\Omega}_{ij} \cup \Gamma_{ij}^{(0)}$. The largest, solid circular circles denote the gluing sets $\Gamma_{ij}^{(i)}$ and $\Gamma_{ij}^{(j)}$. The boldest solid curve with circular symbols is the entire discontinuous boundary set.

Consider a nonpassable boundary formed by two nonpassable sub-boundaries and a gluing point, expressed by

$$\overline{\partial\Omega}_{ij} = \widetilde{\partial\Omega}_{ij} \cup \Gamma_{ij}^{(0)} \cup \widehat{\partial\Omega}_{ij}. \tag{5.19}$$

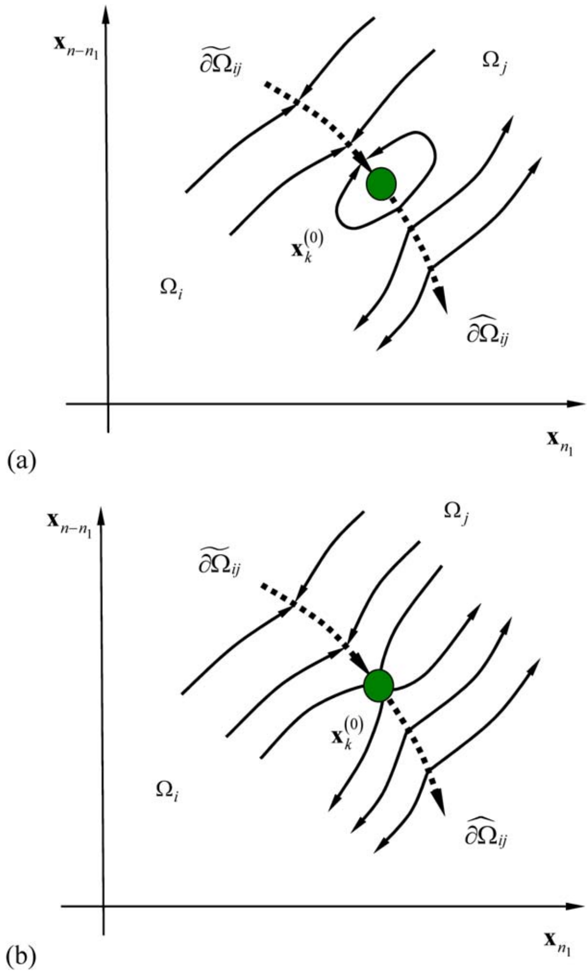


Figure 5.7. Phase portraits for (a) parabolic, (b) hyperbolic flows near the nonpassable boundary $\overline{\partial\Omega}_{ij} = \widetilde{\partial\Omega}_{ij} \cup \Gamma_{ij}^{(0)} \cup \widehat{\partial\Omega}_{ij}$. The largest, solid circular circles denote the gluing sets $\Gamma_{ij}^{(i)}$ and $\Gamma_{ij}^{(j)}$. The dashed curve is the nonpassable boundary.

For a passable boundary involving the nonpassable sub-boundary of the first and second kinds, the boundaries are formed as

$$\overleftrightarrow{\partial\Omega}_{ij} = \overrightarrow{\partial\Omega}_{ij} \cup \underbrace{\Gamma_{ij}^{(j)} \cup \widetilde{\partial\Omega}_{ij} \cup \Gamma_{ij}^{(i)}}_{\text{sliding}} \cup \overleftarrow{\partial\Omega}_{ij},$$

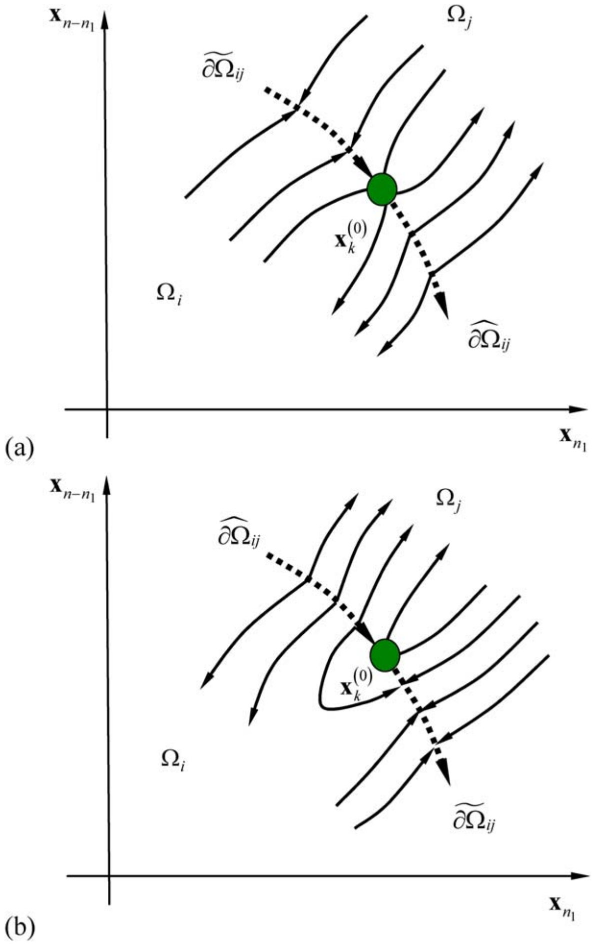


Figure 5.8. Phase portraits for (a), (b) inversed C-flows near the nonpassable boundary $\overline{\partial\Omega}_{ij} = \widetilde{\partial\Omega}_{ij} \cup \Gamma_{ij}^{(0)} \cup \widehat{\partial\Omega}_{ij}$. The largest, solid circular circles denote the gluing sets $\Gamma_{ij}^{(i)}$ and $\Gamma_{ij}^{(j)}$. The dashed curve is the nonpassable boundary.

$$\overleftrightarrow{\partial\Omega}_{ij} = \overrightarrow{\partial\Omega}_{ij} \cup \underbrace{\Gamma_{ij}^{(i)} \cup \widehat{\partial\Omega}_{ij} \cup \Gamma_{ij}^{(j)}}_{\text{outflow}} \cup \overleftarrow{\partial\Omega}_{ij}, \tag{5.20}$$

$$\overleftrightarrow{\partial\Omega}_{ij} = \overrightarrow{\partial\Omega}_{ij} \cup \underbrace{\Gamma_{ij}^{(j)} \cup \overline{\partial\Omega}_{ij} \cup \Gamma_{ij}^{(j)}}_{\text{slinding and outflow}} \cup \overleftarrow{\partial\Omega}_{ij}.$$

As in Eq. (5.18), the generalized boundary with nonpassable sub-boundaries can be developed. To demonstrate the discontinuous boundary including the nonpass-

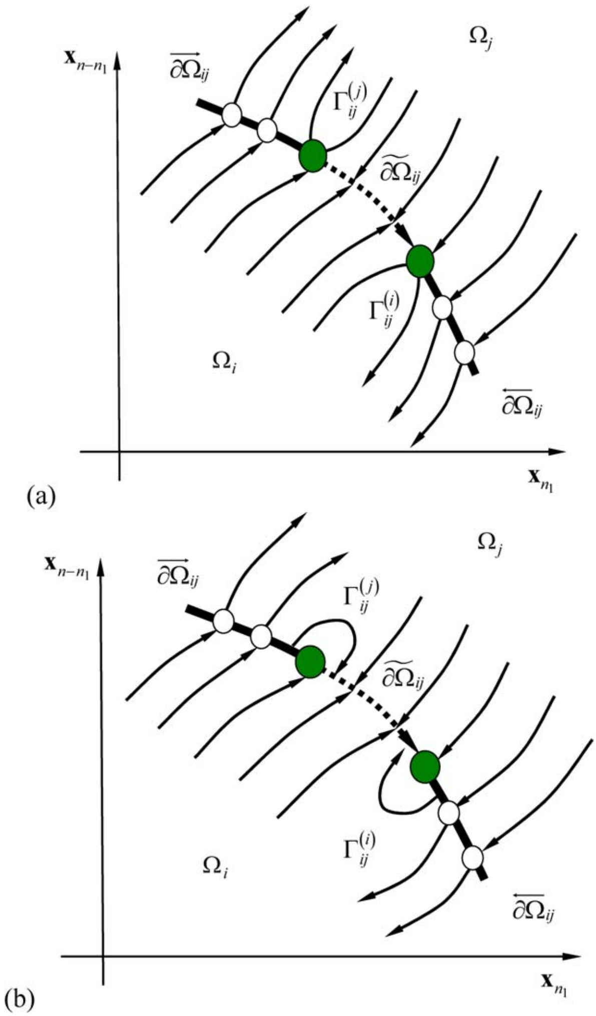


Figure 5.9. Phase portraits near the passable boundary with a sliding nonpassable sub-boundary: (a) semi-hyperbolic flows, and (b) semi-parabolic flows. The largest, solid circular circles denote the gluing sets $\Gamma_{ij}^{(i)}$ and $\Gamma_{ij}^{(j)}$. The boldest solid curve with circular symbols plus the dashed bold curve is the entire discontinuous boundary set. The dashed curve is the nonpassable boundary.

able boundary, the phase portraits near the nonpassable boundary are sketched in Figs. 5.7(a), (b) and 5.8(a), (b). The nonpassable boundaries of the first and second kinds are connected by a gluing point $\mathbf{x}_k^{(0)} \in \Gamma_{ij}$. The largest, solid circular circle is the gluing set $\mathbf{x}_k^{(0)} \in \Gamma_{ij}^{(0)}$. The dashed curve is the nonpassable boundary.

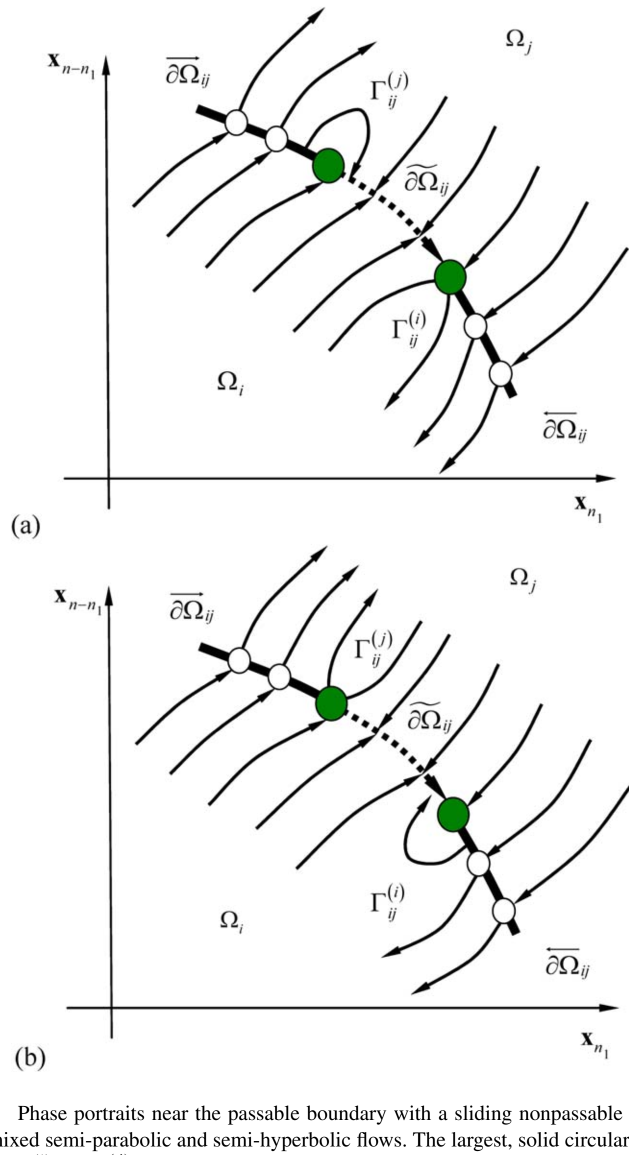


Figure 5.10. Phase portraits near the passable boundary with a sliding nonpassable sub-boundary: (a) and (b) mixed semi-parabolic and semi-hyperbolic flows. The largest, solid circular circles denote the gluing sets $\Gamma_{ij}^{(i)}$ and $\Gamma_{ij}^{(j)}$. The boldest solid curve with circular symbols plus the dashed bold curve is the entire discontinuous boundary set. The dashed curve is the nonpassable boundary.

The parabolic, hyperbolic and inversed C-motions exist in the neighborhood of the gluing point $\mathbf{x}_k^{(0)}$. Similarly, the phase portraits near the passable discontinuous boundary sets with the nonpassable boundary of the first kind are depicted in Figs. 5.9(a), (b) and 5.10(a), (b). Two semi-gluing points are used to connect the nonpassable boundary and semi-passable boundaries. In the neighborhood of the semi-gluing points, the hyperbolicity of the flows to the semi-gluing point is similar to the one for the gluing points, and either semi-hyperbolic or semi-parabolic behaviors of flows in such a neighborhood exist as well. Such phenomena exist in the vicinity of the passable boundary with the nonpassable boundary of the second kind.

5.4. Real flows

Consider a dynamic system consisting of n subdynamic systems in a universal domain $\Omega \subset \Re^n$. The universal domain is divided into m accessible subdomains Ω_i, and the union of all the accessible subdomains $\bigcup_{i=1}^n \Omega_i$ and the universal domain $\Omega = \bigcup_{i=1}^n \Omega_i \cup \Omega_0$, as shown in Fig. 2.1. Ω_0 is the union of the inaccessible domains. For an accessible domain Ω_i, the vector field $\mathbf{F}^{(i)}(\mathbf{x}_i^{(i)}, t, \boldsymbol{\mu}_i)$ is defined on such a domain. The dynamical system in Eq. (2.1) just satisfies the aforementioned condition. The corresponding flow given by such a dynamical system is called a real flow. The more strictly mathematical definition is given as follows.

DEFINITION 5.7. The C^{r+1}-continuous flow $\mathbf{x}_i^{(i)}(t) = \boldsymbol{\Phi}^{(i)}(\mathbf{x}_i^{(i)}(t_0), t, \boldsymbol{\mu}_i)$ is a real flow in the ith open subdomain Ω_i, which is determined by a C^r-continuous system ($r \geqslant 1$) on Ω_i in a form of

$$\dot{\mathbf{x}}_i^{(i)} \equiv \mathbf{F}^{(i)}\big(\mathbf{x}_i^{(i)}, t, \boldsymbol{\mu}_i\big) \in \Re^2, \quad \mathbf{x}_i^{(i)} = \big(x_{i1}^{(i)}, x_{i2}^{(i)}, \ldots, x_{in}^{(i)}\big)^{\mathrm{T}} \in \Omega_i, \tag{5.21}$$

with the initial condition

$$\mathbf{x}_i^{(i)}(t_0) = \boldsymbol{\Phi}^{(i)}\big(\mathbf{x}_i^{(i)}(t_0), t_0, \boldsymbol{\mu}_i\big). \tag{5.22}$$

Notice that time is t and $\dot{\mathbf{x}}_i^{(i)} = \mathrm{d}\mathbf{x}_i^{(i)}/\mathrm{d}t$. The vector field $\mathbf{F}^{(i)}(\mathbf{x}_i^{(i)}, t, \boldsymbol{\mu}_i) \equiv \mathbf{F}_i^{(i)}(t)$ in the domain Ω_i has system parameter vectors $\boldsymbol{\mu}_i = (\mu_i^{(1)}, \mu_i^{(2)}, \ldots, \mu_i^{(l)})^{\mathrm{T}} \in \Re^l$ is C^r-continuous ($r \geqslant 1$) in $\mathbf{x}$ and for all time t. $\boldsymbol{\Phi}^{(i)}(\mathbf{x}_i^{(i)}(t), t_0, \boldsymbol{\mu}_i) \equiv \boldsymbol{\Phi}_i^{(i)}(t)$. $\mathbf{x}_i^{(t)}(t)$ denotes the flow in the ith subdomain Ω_i, governed by a dynamical system defined on the ith subdomain Ω_i. As in Luo (2005a), the hypotheses for the theory of nonsmooth dynamical systems in Eq. (5.21) become as follows:

A1. The switching between two adjacent subsystems possesses time-continuity.

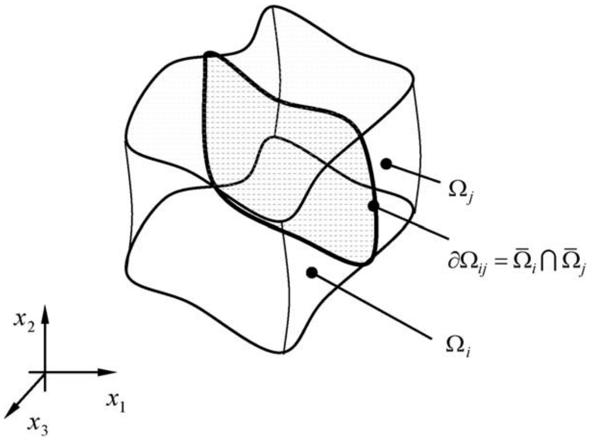


Figure 5.11. A 3-D subdomains Ω_i and Ω_j with the boundary $\partial\Omega_{ij}$.

A2. For an unbounded, accessible subdomain Ω_i, there is an open domain $D_i \subset \Omega_i$. The vector field and flow on D_i are bounded, i.e.,

$$\begin{aligned} &\left\|\mathbf{F}_i^{(i)}\right\| \leqslant K_1 \text{ (const)} \quad \text{and} \\ &\left\|\mathbf{\Phi}_i^{(i)}\right\| \leqslant K_2 \text{ (const)} \quad \text{on } D_i \text{ for } t \in [0, \infty). \end{aligned} \tag{5.23}$$

A3. For a bounded, accessible domain Ω_i, there is an open domain $D_i \subseteq \Omega_i$. The vector field on D_i is bounded but the corresponding flow may be unbounded, i.e.,

$$\begin{aligned} &\left\|\mathbf{F}_i^{(i)}\right\| \leqslant K_1 \text{ (const)} \quad \text{and} \\ &\left\|\mathbf{\Phi}_i^{(i)}\right\| < \infty \quad \text{on } D_i \text{ for } t \in [0, \infty). \end{aligned} \tag{5.24}$$

Consider a boundary set of any two accessible subdomains, formed by the intersection of the closed subdomains, i.e., $\partial\Omega_{ij} = \bar{\Omega}_i \cap \bar{\Omega}_j$ ($i, j \in \{1, 2, \ldots, n\}$, $j \neq i$). A 3-D sketch for two adjacent subdomains with the corresponding boundary is shown in Fig. 5.11. In Section 5.2, the δ-domain of singular points on the separation boundary was introduced to investigate the dynamics of the singular points. To describe the dynamical characteristics of flows near the separation boundary in nonsmooth dynamical systems, the δ-layer of the boundary $\partial\Omega_{ij}$ is introduced that is the neighborhood of $\partial\Omega_{ij}$ in phase plane. Such δ-sublayers in the neighborhood of $\partial\Omega_{ij}$ are illustrated in Fig. 5.12. The δ-sublayers $\delta\Omega_\alpha$ ($\alpha \in \{i, j\}$) are expressed by the shaded areas. The following mathematical description of the δ-layer of the boundary $\partial\Omega_{ij}$ is given by the following definition.

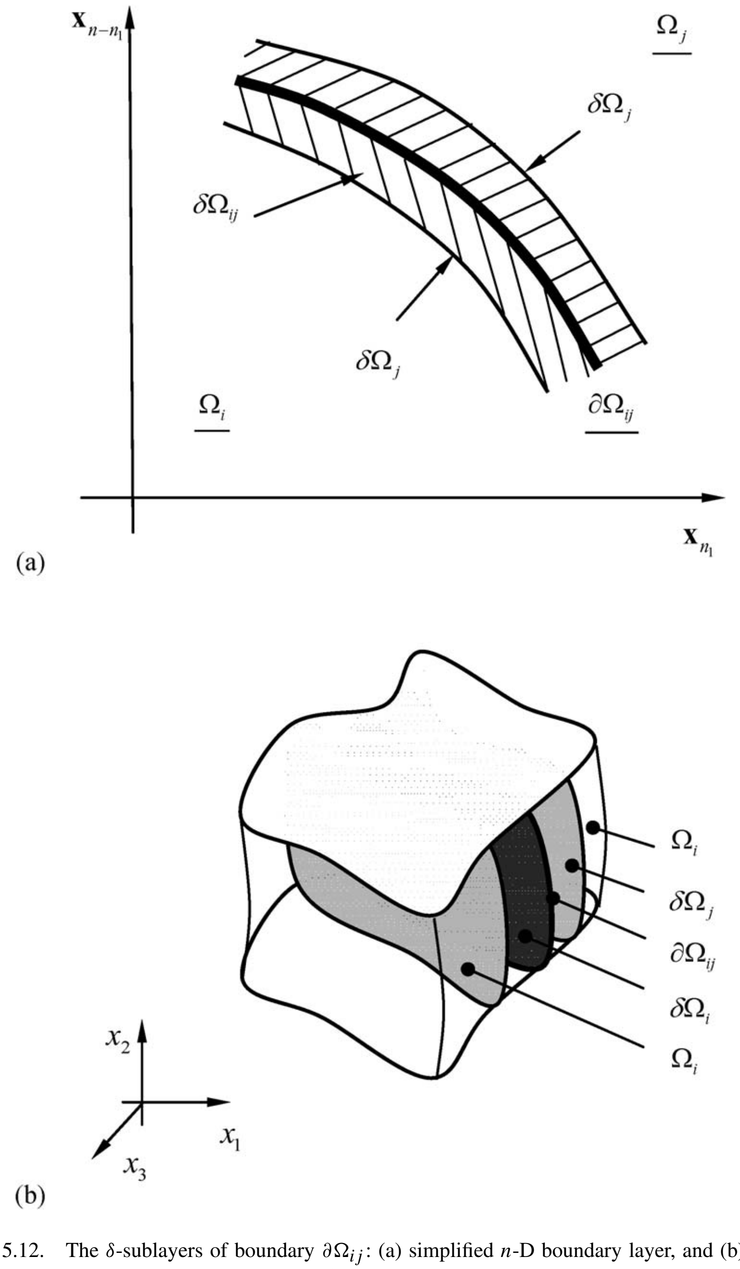


Figure 5.12. The δ-sublayers of boundary $\partial\Omega_{ij}$: (a) simplified n-D boundary layer, and (b) a 3-D view of boundary layer.

DEFINITION 5.8. The δ-layer of the boundary $\partial\Omega_{ij}$ is defined as

$$\delta\Omega_{ij} \equiv \delta\Omega_i \cup \delta\Omega_j \cup \partial\Omega_{ij}, \tag{5.25}$$

while the δ-sublayer in Ω_α, $\alpha \in \{i, j\}$, is defined as

$$\delta\Omega_\alpha = \left\{\mathbf{x}_\alpha^{(\alpha)} \in \Omega_\alpha \,\middle|\, \text{for given } \delta_\alpha > 0, \left\|\mathbf{x}_\alpha^{(\alpha)} - \mathbf{x}_{\alpha\beta}^{(0)}\right\| < \delta_\alpha, \mathbf{x}_{\alpha\beta}^{(0)} \in \partial\Omega_{ij}\right\}. \tag{5.26}$$

For forming a separation boundary, except for the semi-passable and nonpassable boundaries, the gluing point connecting two portions of separatrix on the boundary is very important, and on the new boundary the flows possess two different vector characteristics. As in Section 5.1, the gluing points are re-defined through real flows to develop the imaginary flows of such dynamical systems later.

DEFINITION 5.9. A singular set on the boundary $\partial\Omega_{ij}$

$$\Gamma_{ij} = \left\{\mathbf{x}_{(k)}^{(0)} \in \partial\Omega_{ij} \,\middle|\, \mathbf{x}_\alpha^{(\alpha)} \in \Omega_\alpha, \lim_{\mathbf{x}_\alpha^{(\alpha)} \to \mathbf{x}_{(k)}^{(0)}} \left(\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}_\alpha^{(\alpha)}\right) = 0, \right.$$
$$\left. \alpha \in \{i, j\},\ k \in \mathbb{N}\right\} \tag{5.27}$$

is termed the gluing point set, and $\mathbb{N}$ is the natural number set.

DEFINITION 5.10. A singular set on the boundary $\partial\Omega_{ij}$

$$\Gamma_{ij}^{(\alpha)} = \left\{\mathbf{x}_{(k)}^{(0)} \in \partial\Omega_{ij} \,\middle|\, \mathbf{x}_\alpha^{(\alpha)} \in \Omega_\alpha, \lim_{\mathbf{x}_\alpha^{(\alpha)} \to \mathbf{x}_{(k)}^{(0)}} \left(\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \dot{\mathbf{x}}_\alpha^{(\alpha)}\right) = 0, \right.$$
$$\left. \alpha = \{i \text{ or } j\},\ k \in \mathbb{N}\right\} \subset \Gamma_{ij} \tag{5.28}$$

is termed the input or output, semi-gluing, singular set on the boundary on the side of Ω_α.

The above definition of $\Gamma_{ij}^{(\alpha)}$ indicates the switching of the flow direction at the singular point on the side of Ω_α.

DEFINITION 5.11. A singular set on the boundary $\partial\Omega_{ij}$

$$\Gamma_{ij}^{(0)} = \left\{\mathbf{x}_k^{(0)} \in \partial\Omega_{ij} \,\middle|\, \mathbf{x}_\alpha^{(\alpha)} \in \Omega_\alpha, \lim_{\mathbf{x}_\alpha^{(\alpha)} \to \mathbf{x}_k^{(0)}} \left(\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \dot{\mathbf{x}}_\alpha^{(\alpha)}\right) = 0, \right.$$
$$\left. \alpha = \{i \text{ and } j\},\ k \in \mathbb{N}\right\} \subset \Gamma_{ij} \tag{5.29}$$

is termed the full-gluing, singular set.

The foregoing definition of $\Gamma_{ij}^{(0)}$ indicates the switching of the flow direction at the singular point on both sides of Ω_α. The gluing point set is $\Gamma_{ij} = \Gamma_{ij}^{(i)} \cup \Gamma_{ij}^{(j)} \cup \Gamma_{ij}^{(0)}$. The gluing sets connecting the semi-passable and nonpassable boundaries will form a new separation boundary. The definitions and theorems for semi-passable and nonpassable boundaries were presented in Luo (2005a), which are used in the further discussion. Therefore, to compare with imagery flows in next section, they are briefly stated through the real flow.

DEFINITION 5.12. For a discontinuous dynamical system in Eq. (5.21), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. Suppose $\mathbf{x}_i^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}_j^{(j)}(t_{m+})$. The real flow $\mathbf{x}_\alpha^{(\alpha)}(t)$ $(\alpha \in \{i, j\})$ from the domain Ω_i to Ω_j is semi-passable to the nonempty boundary set $\partial\Omega_{ij}$ if the flow $\mathbf{x}_\alpha^{(\alpha)}(t)$ possesses the following properties:

$$\begin{aligned} &\text{either} \quad \begin{cases} \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}_i^{(i)}(t_{m-\varepsilon})\right] > 0 \\ \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \left[\mathbf{x}_j^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] > 0 \end{cases} \\ &\qquad\quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ &\text{or} \quad \begin{cases} \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \left[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}_i^{(i)}(t_{m-\varepsilon})\right] < 0 \\ \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \left[\mathbf{x}_j^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\right] < 0 \end{cases} \\ &\qquad\quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{aligned} \tag{5.30}$$

THEOREM 5.3. *For a discontinuous dynamical system in Eq. (5.21), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. Suppose $\mathbf{x}_i^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}_j^{(j)}(t_{m+})$. Both $\mathbf{x}_i^{(i)}(t)$ and $\mathbf{x}_j^{(j)}(t)$ are $C^r_{[t_{m-\varepsilon}, t_m)}$- and $C^r_{(t_m, t_{m+\varepsilon}]}$-continuous $(r \geqslant 2)$ for time t, respectively, and $\|\mathrm{d}^r \mathbf{x}_\alpha^{(\alpha)}/\mathrm{d}t^r\| < \infty$ $(\alpha \in \{i, j\})$. The real flow $\mathbf{x}_\alpha^{(\alpha)}(t)$ $(\alpha \in \{i, j\})$ from the domain Ω_i to Ω_j is semi-passable to the nonempty boundary set $\partial\Omega_{ij}$ iff*

$$\begin{aligned} &\textit{either} \quad \begin{cases} \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}_i^{(i)}(t_{m-}) > 0 \\ \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}_j^{(j)}(t_{m+}) > 0 \end{cases} \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ &\textit{or} \quad \begin{cases} \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}_i^{(i)}(t_{m-}) < 0 \\ \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}_j^{(j)}(t_{m+}) < 0 \end{cases} \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{aligned} \tag{5.31}$$

THEOREM 5.4. *For a discontinuous dynamical system in Eq. (5.21), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. Suppose $\mathbf{x}_i^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}_j^{(j)}(t_{m+})$. Both $\mathbf{F}_i^{(i)}(t)$ and*

$\mathbf{F}_j^{(j)}(t)$ *are* $C^r_{[t_{m-\varepsilon},t_m)}$- *and* $C^r_{(t_m,t_{m+\varepsilon}]}$-*continuous* ($r \geqslant 1$) *for time t, respectively, and* $\|\mathrm{d}^{r+1}\mathbf{x}_\alpha^{(\alpha)}/\mathrm{d}t^{r+1}\| < \infty$ ($\alpha \in \{i, j\}$). *The real flow* $\mathbf{x}_\alpha^{(\alpha)}(t)$ ($\alpha \in \{i, j\}$) *from the domain* Ω_i *to* Ω_j *is semi-passable to the nonempty boundary set* $\partial\Omega_{ij}$ *iff*

$$\begin{aligned} &\textit{either} \quad \begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}_i^{(i)}(t_{m-}) > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}_j^{(j)}(t_{m+}) > 0 \end{cases} \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ &\textit{or} \quad \begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}_i^{(i)}(t_{m-}) < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}_j^{(j)}(t_{m+}) < 0 \end{cases} \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i \end{aligned} \tag{5.32}$$

where $\mathbf{F}_i^{(i)}(t_{m-}) \equiv \mathbf{F}^{(i)}(\mathbf{x}_i^{(i)}, t_{m-}, \boldsymbol{\mu}_i)$ *and* $\mathbf{F}_j^{(j)}(t_{m+}) = \mathbf{F}^{(j)}(\mathbf{x}_j^{(j)}, t_{m+}, \boldsymbol{\mu}_j)$.

DEFINITION 5.13. For a discontinuous dynamical system in Eq. (5.21), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at t_m. For an arbitrarily small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_m)$. Suppose $\mathbf{x}_\alpha^{(\alpha)}(t_{m-}) = \mathbf{x}_m$ ($\alpha \in \{i, j\}$). The nonempty boundary $\partial\Omega_{ij}$ is the non-passable boundary of the first kind to the real flow $\mathbf{x}_\alpha^{(\alpha)}$ ($\alpha \in \{i, j\}$) if the following condition is satisfied:

$$\big\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}_i^{(i)}(t_{m-\varepsilon})\big]\big\}\big\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathbf{x}^{(0)}(t_{m-\varepsilon}) - \mathbf{x}_j^{(j)}(t_{m-\varepsilon})\big]\big\} < 0. \tag{5.33}$$

DEFINITION 5.14. For a discontinuous dynamical system in Eq. (5.21), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at t_m. For an arbitrarily small $\varepsilon > 0$, there is a time interval $(t_m, t_{m+\varepsilon}]$. Suppose $\mathbf{x}_\alpha^{(\alpha)}(t_{m+}) = \mathbf{x}_m$ ($\alpha \in \{i, j\}$). The nonempty boundary $\partial\Omega_{ij}$ is the non-passable boundary of the second kind to the real flow $\mathbf{x}_\alpha^{(\alpha)}$ ($\alpha \in \{i, j\}$) if the following condition is satisfied:

$$\big\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathbf{x}_i^{(i)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\big]\big\}\big\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathbf{x}_j^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(0)}(t_{m+\varepsilon})\big]\big\} < 0. \tag{5.34}$$

THEOREM 5.5. *For a discontinuous dynamical system in Eq.* (5.21), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *at* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there is a time interval* $[t_{m-\varepsilon}, t_m)$. *Suppose* $\mathbf{x}_\alpha^{(\alpha)}(t_{m-}) = \mathbf{x}_m$ ($\alpha \in \{i, j\}$). *The real flow* $\mathbf{x}_\alpha^{(\alpha)}(t)$ *is* $C^r_{[t_{m-\varepsilon},t_m)}$-*continuous* ($r \geqslant 2$) *for time t and* $\|\mathrm{d}^r\mathbf{x}_\alpha^{(\alpha)}/\mathrm{d}t^r\| < \infty$. *The nonempty boundary* $\partial\Omega_{ij}$ *is the nonpassable boundary of the first kind to the real flow* $\mathbf{x}_\alpha^{(\alpha)}$ ($\alpha \in \{i, j\}$) *iff*

$$\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}_i^{(i)}(t_{m-})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}_j^{(j)}(t_{m-})\big] < 0. \tag{5.35}$$

THEOREM 5.6. *For a discontinuous dynamical system in Eq.* (5.21), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *at* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there is a time interval*

$[t_{m-\varepsilon}, t_m)$. *Suppose* $\mathbf{x}_\alpha^{(\alpha)}(t_{m-}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$. *The real vector field* $\mathbf{F}_\alpha^{(\alpha)}(t)$ *is* $C^r_{[t_{m-\varepsilon}, t_m)}$*-continuous* $(r \geqslant 1)$ *for time t and* $\|\mathrm{d}^{r+1}\mathbf{x}_\alpha^{(\alpha)}/\mathrm{d}t^{r+1}\| < \infty$. *The nonempty boundary* $\partial\Omega_{ij}$ *is the nonpassable boundary of the second kind to the real flow* $\mathbf{x}_\alpha^{(\alpha)}$ *iff*

$$\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}_i^{(i)}(t_{m-})\right]\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}_j^{(j)}(t_{m-})\right] < 0 \tag{5.36}$$

where $\mathbf{F}_\alpha^{(\alpha)}(t_{m-}) \triangleq \mathbf{F}^{(\alpha)}(\mathbf{x}_\alpha^{(\alpha)}, t_{m-}, \boldsymbol{\mu}_\alpha)$ $(\alpha \in \{i, j\})$.

THEOREM 5.7. *For a discontinuous dynamical system in Eq. (5.21),* $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *at* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there is a time interval* $(t_m, t_{m+\varepsilon}]$. *Suppose* $\mathbf{x}_\alpha^{(\alpha)}(t_{m+}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$. *The real flow* $\mathbf{x}_\alpha^{(\alpha)}$ *is* $C^r_{(t_m, t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 2)$ *for time t and* $\|\mathrm{d}^r\mathbf{x}_\alpha^{(\alpha)}/\mathrm{d}t^r\| < \infty$. *The nonempty boundary set* $\partial\Omega_{ij}$ *is a nonpassable boundary of the second kind to the real flow* $\mathbf{x}_\alpha^{(\alpha)}$ $(\alpha \in \{i, j\})$ *iff*

$$\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}_i^{(i)}(t_{m+})\right]\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}_j^{(j)}(t_{m+})\right] < 0. \tag{5.37}$$

THEOREM 5.8. *For a discontinuous dynamical system in Eq. (5.21),* $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *at* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there is a time interval* $(t_m, t_{m+\varepsilon}]$. *Suppose* $\mathbf{x}_\alpha^{(\alpha)}(t_{m+}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$. *The real vector field* $\mathbf{F}_\alpha^{(\alpha)}(t)$ *is* $C^r_{(t_m, t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 1)$ *for time t and* $\|\mathrm{d}^{r+1}\mathbf{x}_\alpha^{(\alpha)}/\mathrm{d}t^{r+1}\| < \infty$. *The nonempty boundary set* $\partial\Omega_{ij}$ *is a nonpassable boundary of the second kind to the real flow* $\mathbf{x}_\alpha^{(\alpha)}$ $(\alpha \in \{i, j\})$ *iff*

$$\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}_i^{(i)}(t_{m+})\right]\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}_j^{(j)}(t_{m+})\right] < 0 \tag{5.38}$$

where $\mathbf{F}_\alpha^{(\alpha)}(t_{m+}) \triangleq \mathbf{F}^{(\alpha)}(\mathbf{x}_\alpha^{(\alpha)}, t_{m+}, \boldsymbol{\mu}_\alpha)$.

The other definitions and theorems in Chapters 2–4 can be expressed by the real flows in a similar fashion.

5.5. Imaginary flows

The above definitions and theorems of semi-passable and nonpassable boundaries are based on the real flows of the nonsmooth dynamical system in Eq. (5.21). The real flow $\mathbf{x}_i^{(i)}(t)$ in Ω_i is governed by a dynamical system on its own domain. However, another flow $\mathbf{x}_i^{(j)}$ in Ω_i is governed by a dynamical system defined on the jth subdomain Ω_j, which is of great interest herein. This kind of flow is called an *imaginary flow* because the flow is not determined by the dynamical system on its own domain. To understand further the dynamical behavior of the nonsmooth

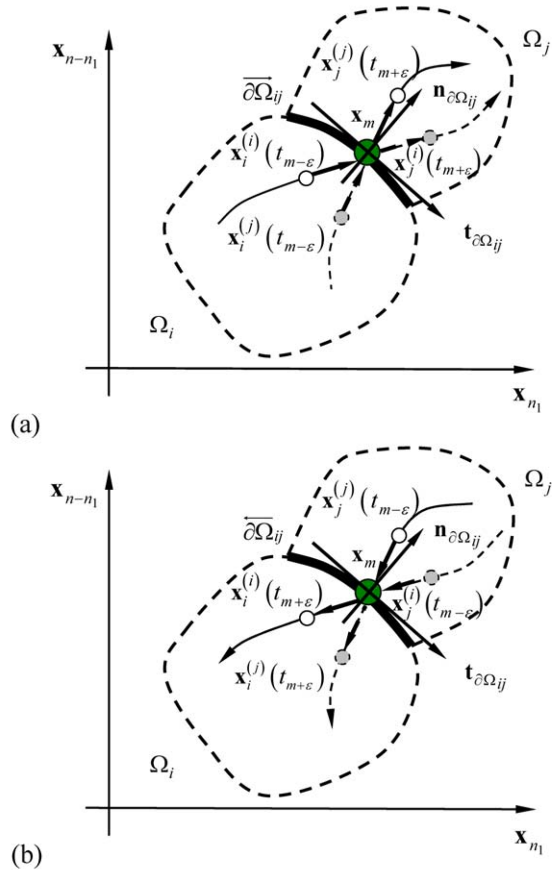


Figure 5.13. Real and imaginary flows in the neighborhood of (a), (b) the semi-passable boundary. The boundary point $\mathbf{x}_m$ is at time t_m. The solid and dashed curves represent the real and imaginary flows.

dynamical system, it is necessary to introduce the imaginary flows. Let the jth imaginary flow in the ith domain Ω_i be a flow in Ω_i governed by the dynamical system defined on the jth subdomain Ω_j. The flow is not a real one governed by the discontinuous dynamical system, thus this flow is also termed the *imaginary* flow in this sense. In additions, the two subdomains can be either adjacent or separable. The mathematical definition of imaginary flows is as follows.

DEFINITION 5.15. The C^{r+1} ($r \geqslant 1$)-continuous flow $\mathbf{x}_i^{(j)}(t)$ is termed the jth imaginary flow in the ith open subdomain Ω_i if the flow $\mathbf{x}_i^{(j)}(t)$ is determined by application of a C^r-continuous system, defined on the jth open subdomain Ω_j, to

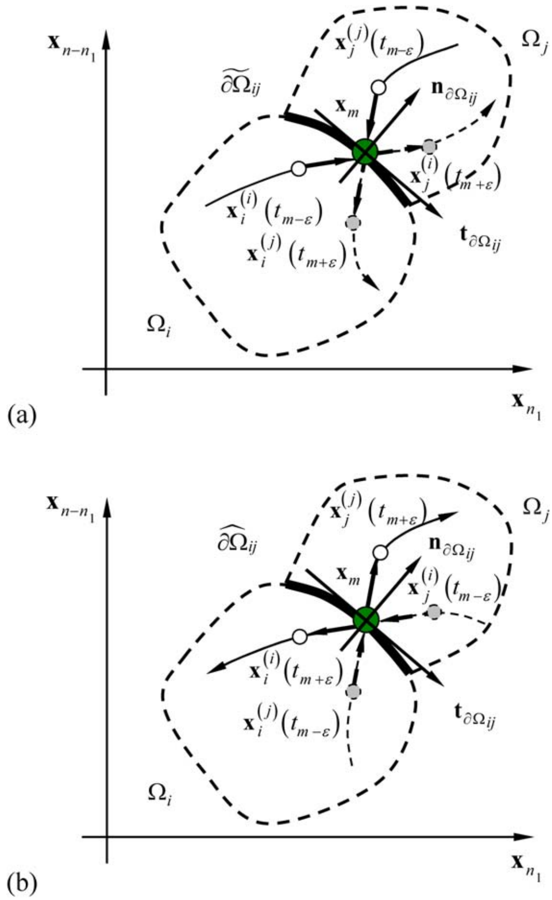


Figure 5.14. Real and imaginary flows in the neighborhood of (a) the sink boundary, and (b) the source boundary. The boundary point $\mathbf{x}_m$ is at time t_m. The solid and dashed curves represent the real and imaginary flows.

the ith open subdomain Ω_i, i.e.,

$$\dot{\mathbf{x}}_i^{(j)} \equiv \mathbf{F}^{(j)}\big(\mathbf{x}_i^{(j)}, t, \boldsymbol{\mu}_j\big) \in \Re^2, \quad \mathbf{x}_i^{(j)} = \big(x_{1i}^{(j)}, x_{2i}^{(j)}, \ldots, x_{ni}^{(j)}\big)^{\mathrm{T}} \in \Omega_i, \tag{5.39}$$

with the initial conditions

$$\mathbf{x}_i^{(j)}(t_0) = \boldsymbol{\Phi}^{(j)}\big(\mathbf{x}_i^{(j)}(t_0), t_0, \boldsymbol{\mu}_j\big). \tag{5.40}$$

To demonstrate the above concept, the real and imaginary flows in two adjacent subdomains are illustrated in Figs. 5.13 and 5.14 for the semi-passable boundary, the sink boundary and the source boundary. The boundary point $\mathbf{x}_m$ is at time t_m, and $\mathbf{x}_{\alpha}^{(\alpha)}(t_m) = \mathbf{x}_m = \mathbf{x}_{\beta}^{(\alpha)}(t_m)$ for $\alpha, \beta \in \{i, j\}$ and $\alpha \neq \beta$. The real flows $\mathbf{x}_{\alpha}^{(\alpha)}(t)$ are represented by solid curves. The imaginary flows $\mathbf{x}_{\alpha}^{(\beta)}(t)$ in the domain Ω_α are depicted by dashed curves. $\mathbf{x}_{\alpha}^{(\alpha)}(t_{m\pm\varepsilon})$ and $\mathbf{x}_{\alpha}^{(\beta)}(t_{m\pm\varepsilon})$ in the δ-layer are the values of the *real* and *imaginary* flows at $t_{m\pm\varepsilon} = t_m \pm \varepsilon$ for an arbitrary $\varepsilon > 0$. As $\varepsilon \to 0$, $\{\mathbf{x}_{\alpha}^{(\alpha)}(t_{m\pm\varepsilon}), \mathbf{x}_{\alpha}^{(\beta)}(t_{m\pm\varepsilon})\} \to \mathbf{x}_m$. From the foregoing definition, the flow $\mathbf{x}_{\alpha}^{(\alpha)}(t) \cup \mathbf{x}_{\beta}^{(\alpha)}(t)$ gives a continuous flow in the two subdomains Ω_i and Ω_j plus the boundary $\partial\Omega_{ij}$. Therefore, Definitions 5.12–5.14 for the real flows near the semi-passable, sink and source boundaries are applicable to the imaginary flows. Similarly, Theorems 5.3–5.8 are applicable to the imaginary flows.

THEOREM 5.9. *For a discontinuous dynamical system in Eq.* (5.21), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. *For arbitrarily small* $\varepsilon > 0$, *there are two time intervals* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$ *in the* δ*-layer of* $\partial\Omega_{ij}$. *Suppose* $\{\mathbf{x}_{\alpha}^{(\alpha)}(t_{m-}), \mathbf{x}_{\beta}^{(\beta)}(t_{m+})\} = \mathbf{x}_m$, $\{\mathbf{x}_{\alpha}^{(\beta)}(t_{m-}), \mathbf{x}_{\beta}^{(\alpha)}(t_{m+})\} = \mathbf{x}_m$ $(\alpha, \beta \in \{i, j\}$ *and* $\alpha \neq \beta)$ *hold, the real and imaginary flows* $\{\mathbf{x}_{\alpha}^{(\alpha)}(t), \mathbf{x}_{\beta}^{(\alpha)}(t)\}$ *and* $\{\mathbf{x}_{\alpha}^{(\beta)}(t), \mathbf{x}_{\beta}^{(\beta)}(t)\}$ *are* $C^r_{[t_{m-\varepsilon}, t_m)}$*- and* $C^r_{(t_m, t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 2)$ *for time* t, *respectively, and* $\|\mathrm{d}^r\mathbf{x}_{\alpha}^{(\beta)}/\mathrm{d}t^r\| < \infty$ $(\alpha, \beta \in \{i, j\})$.

(i) *The boundary set* $\partial\Omega_{\alpha\beta}$ *to the real and imaginary flows is semi-passable* $\overrightarrow{\partial\Omega}_{\alpha\beta}$ *iff*

$$\begin{aligned}
&\big[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}_{\alpha}^{(\alpha)}(t_{m-})\big]\big[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}_{\alpha}^{(\beta)}(t_{m-})\big] > 0 \quad \text{or} \\
&\big[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}_{\beta}^{(\alpha)}(t_{m+})\big]\big[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}_{\alpha}^{(\beta)}(t_{m+})\big] > 0.
\end{aligned} \tag{5.41}$$

(ii) *The boundary set* $\partial\Omega_{\alpha\beta}$ *to the real and imaginary flows is nonpassable of the first kind (sink boundary)* $\widetilde{\partial\Omega}_{\alpha\beta}$ *iff*

$$\begin{aligned}
&\big[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}_{\alpha}^{(\alpha)}(t_{m-})\big]\big[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}_{\alpha}^{(\beta)}(t_{m+})\big] < 0 \quad \text{or} \\
&\big[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}_{\beta}^{(\alpha)}(t_{m+})\big]\big[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}_{\alpha}^{(\beta)}(t_{m+})\big] < 0.
\end{aligned} \tag{5.42}$$

(iii) *The boundary set* $\partial\Omega_{\alpha\beta}$ *to the real and imaginary flows is nonpassable of the second kind (source boundary)* $\widehat{\partial\Omega}_{\alpha\beta}$ *iff*

$$\begin{aligned}
&\big[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}_{\alpha}^{(\alpha)}(t_{m+})\big]\big[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}_{\alpha}^{(\beta)}(t_{m-})\big] < 0 \quad \text{or} \\
&\big[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}_{\beta}^{(\alpha)}(t_{m-})\big]\big[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}_{\alpha}^{(\beta)}(t_{m-})\big] < 0.
\end{aligned} \tag{5.43}$$

PROOF. Consider all points $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \overrightarrow{\partial\Omega}_{\alpha\beta}$ in the δ-layer, and application of Theorem 5.3 to the real and imaginary flows gives the necessary and sufficient conditions as

$$\text{either}\quad \begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\alpha)}_{\alpha}(t_{m-}) > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\beta)}_{\beta}(t_{m+}) > 0 \end{cases} \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_{\beta}$$

$$\text{or}\quad \begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\alpha)}_{\alpha}(t_{m-}) < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\beta)}_{\beta}(t_{m+}) < 0 \end{cases} \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_{\alpha},$$

and

$$\text{either}\quad \begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\beta)}_{\alpha}(t_{m-}) > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\alpha)}_{\beta}(t_{m+}) > 0 \end{cases} \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_{\beta}$$

$$\text{or}\quad \begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\beta)}_{\alpha}(t_{m-}) < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\alpha)}_{\beta}(t_{m+}) < 0 \end{cases} \quad \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_{\alpha}.$$

Therefore, no matter how $\partial\Omega_{\alpha\beta}$ is convex to either Ω_{α} or Ω_{β}, the real flows require

$$\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}_{\alpha}(t_{m-})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}_{\beta}(t_{m+})\big] > 0,$$

and the imaginary flows require

$$\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}_{\beta}(t_{m+})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}_{\alpha}(t_{m-})\big] > 0.$$

Because the real and imaginary flows (i.e., $\mathbf{x}^{(i)}_{i}(t) \cup \mathbf{x}^{(i)}_{j}(t)$ and $\mathbf{x}^{(j)}_{i}(t) \cup \mathbf{x}^{(j)}_{j}(t)$) are $C^{r}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous ($r \geqslant 2$) for time t, the following relations at $t = t_{m\pm}$ show the continuity equations between the real and imaginary flows:

$$\left.\frac{\mathrm{d}^s \mathbf{x}^{(\alpha)}_{\alpha}}{\mathrm{d}t^s}\right|_{t=t_{m-}} = \left.\frac{\mathrm{d}^s \mathbf{x}^{(\alpha)}_{\beta}}{\mathrm{d}t^s}\right|_{t=t_{m+}} \quad \text{and} \quad \left.\frac{\mathrm{d}^s \mathbf{x}^{(\beta)}_{\alpha}}{\mathrm{d}t^s}\right|_{t=t_{m-}} = \left.\frac{\mathrm{d}^s \mathbf{x}^{(\beta)}_{\beta}}{\mathrm{d}t^s}\right|_{t=t_{m+}}$$

for $s \in \{0, 1, \ldots, r\}$ and $\alpha, \beta \in \{i, j\}$ with $\alpha \neq \beta$. From the foregoing equation, the following relations hold:

$$\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}_{\alpha}(t_{m-})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}_{\alpha}(t_{m-})\big] > 0,$$
$$\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}_{\beta}(t_{m+})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}_{\beta}(t_{m+})\big] > 0.$$

Similarly, from Theorems 5.4 and 5.6, the continuity equations between the real and imaginary flows imply the following relations:

$$\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}_{\alpha}(t_{m-})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}_{\alpha}(t_{m+})\big] < 0,$$
$$\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}_{\beta}(t_{m+})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}_{\alpha}(t_{m+})\big] < 0;$$

$$\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}_{\alpha}^{(\alpha)}(t_{m+})\right]\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}_{\alpha}^{(\beta)}(t_{m-})\right] < 0,$$
$$\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}_{\beta}^{(\alpha)}(t_{m-})\right]\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}_{\alpha}^{(\beta)}(t_{m-})\right] < 0.$$

The theorem is proved. □

THEOREM 5.10. *For a discontinuous dynamical system in Eq. (5.21),* $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$ *in the* δ*-layer of* $\partial\Omega_{ij}$. *Suppose* $\{\mathbf{x}_{\alpha}^{(\alpha)}(t_{m-}), \mathbf{x}_{\beta}^{(\beta)}(t_{m+})\} = \mathbf{x}_m$, $\{\mathbf{x}_{\alpha}^{(\beta)}(t_{m-}), \mathbf{x}_{\beta}^{(\alpha)}(t_{m+})\} = \mathbf{x}_m$ $(\alpha, \beta \in \{i, j\}$ *and* $\alpha \neq \beta)$ *hold, the real and imaginary vector fields* $\{\mathbf{F}_{\alpha}^{(\alpha)}(t), \mathbf{F}_{\beta}^{(\alpha)}(t)\}$ *and* $\{\mathbf{F}_{\alpha}^{(\beta)}(t), \mathbf{F}_{\beta}^{(\beta)}(t)\}$ *are* $C^r_{[t_{m-\varepsilon}, t_m)}$*- and* $C^r_{(t_m, t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 1)$ *for time t, respectively, and* $\|\mathrm{d}^{r+1}\mathbf{x}_{\alpha}^{(\beta)}/\mathrm{d}t^{r+1}\| < \infty$ $(\alpha, \beta \in \{i, j\})$.

(i) *The boundary set* $\partial\Omega_{\alpha\beta}$ *to the real and imaginary flows is semi-passable* $\overrightarrow{\partial\Omega}_{\alpha\beta}$ *iff*

$$\begin{aligned}&\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}_{\alpha}^{(\alpha)}(t_{m-})\right]\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}_{\alpha}^{(\beta)}(t_{m-})\right] > 0,\\&\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}_{\beta}^{(\alpha)}(t_{m+})\right]\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}_{\beta}^{(\beta)}(t_{m+})\right] > 0.\end{aligned} \tag{5.44}$$

(ii) *The boundary set* $\partial\Omega_{\alpha\beta}$ *to the real and imaginary flows is nonpassable of the first kind (sink boundary)* $\widetilde{\partial\Omega}_{\alpha\beta}$ *iff*

$$\begin{aligned}&\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}_{\alpha}^{(\alpha)}(t_{m-})\right]\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}_{\alpha}^{(\beta)}(t_{m+})\right] < 0, \quad \textit{or}\\&\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}_{\beta}^{(\alpha)}(t_{m+})\right]\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}_{\alpha}^{(\beta)}(t_{m+})\right] < 0.\end{aligned} \tag{5.45}$$

(iii) *The boundary set* $\partial\Omega_{\alpha\beta}$ *to the real and imaginary flows is nonpassable of the second kind (source boundary)* $\widehat{\partial\Omega}_{\alpha\beta}$ *iff*

$$\begin{aligned}&\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}_{\alpha}^{(\alpha)}(t_{m+})\right]\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}_{\alpha}^{(\beta)}(t_{m-})\right] < 0, \quad \textit{or}\\&\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}_{\beta}^{(\alpha)}(t_{m-})\right]\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}_{\alpha}^{(\beta)}(t_{m-})\right] < 0.\end{aligned} \tag{5.46}$$

PROOF. Application of Definitions 5.8 and 5.14 to Eqs. (5.41)–(5.43) from Theorem 5.9. The theorem is proved. □

LEMMA 5.1. *For a discontinuous system in Eq. (5.1), a point* $\mathbf{x}^{(0)} \in \partial\Omega_{ij}$ *is* (i) *a gluing point* $\mathbf{x}^{(0)} \in \Gamma_{ij}$ *for* $\alpha \in \{i, j\}$, (ii) *a semi-gluing point* $\mathbf{x}^{(0)} \in \Gamma_{ij}^{(\alpha)}$ *for* $\alpha = \{i$ *or* $j\}$, *and* (iii) *a full-gluing point* $\mathbf{x}^{(0)} \in \Gamma_{ij}^{(0)}$ *for* $\alpha = \{i$ *and* $j\}$ *iff in the* *δ-layer of* $\partial\Omega_{ij}$

$$\lim_{\mathbf{x}_{\alpha}^{(\alpha)}(t_{m\pm})\to\mathbf{x}_{(k)}^{(0)}}\left(\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\cdot\mathbf{F}_{\alpha}^{(\alpha)}(t_{m\pm})\right)=0,\qquad \lim_{\mathbf{x}_{\alpha}^{(\alpha)}(t_{m\pm})\to\mathbf{x}_{(k)}^{(0)}}\left(\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\cdot\mathbf{F}_{\alpha}^{(\beta)}(t_{m\mp})\right)=0. \tag{5.47}$$

PROOF. Application of Definitions 5.8 and 5.14 to Definitions 5.9–5.11 yields Eq. (5.47). The lemma is proved. □

In Section 5.3, the formation of the boundary was discussed. However, the necessary and sufficient conditions need to be developed for four basic formations of boundaries in nonsmooth, dynamical systems. From the definitions of the semi-passable, nonpassable boundaries and gluing points, the following theorems are presented for such necessary and sufficient conditions. Through the real and imaginary flows, the passable and nonpassable boundaries are described, which can further be used to discuss the conditions for boundary formation. The critical conditions for the motion switching on the boundary can be determined.

THEOREM 5.11. *For a discontinuous dynamical system in Eq.* (5.21), $\mathbf{x}(t_m)\equiv\mathbf{x}_m\in\partial\Omega_{ij}$ *for* t_m. *For arbitrarily small* $\varepsilon>0$, *there are two time intervals* $[t_{m-\varepsilon},t_m)$ *and* $(t_m,t_{m+\varepsilon}]$ *in the δ-layer of* $\partial\Omega_{ij}$. *Suppose* $\{\mathbf{x}_{\alpha}^{(\alpha)}(t_{m-}),\mathbf{x}_{\beta}^{(\beta)}(t_{m+})\}=\mathbf{x}_m$, $\{\mathbf{x}_{\alpha}^{(\beta)}(t_{m-}),\mathbf{x}_{\beta}^{(\alpha)}(t_{m+})\}=\mathbf{x}_m$ $(\alpha,\beta\in\{i,j\}$ *and* $\alpha\neq\beta)$ *hold, the real and imaginary flows* $\{\mathbf{x}_{\alpha}^{(\alpha)}(t),\mathbf{x}_{\beta}^{(\alpha)}(t)\}$ *and* $\{\mathbf{x}_{\alpha}^{(\beta)}(t),\mathbf{x}_{\beta}^{(\beta)}(t)\}$ *are* $C^r_{[t_{m-\varepsilon},t_m)}$*- and* $C^r_{(t_m,t_{m+\varepsilon}]}$*-continuous* $(r\geqslant 2)$ *for time t, respectively,* $\|\mathrm{d}^r\mathbf{x}_{\alpha}^{(\beta)}/\mathrm{d}t^r\|<\infty$ $(\alpha,\beta\in\{i,j\})$ *and* $m\in\{m_1,m_2,m_3\}$.

(i) *A passable boundary* $(\partial\Omega_{ij}=\overrightarrow{\partial\Omega}_{ij}\cup\Gamma_{ij}\cup\overleftarrow{\partial\Omega}_{ij})$ *exists iff*

$$\begin{aligned}
&\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\cdot\dot{\mathbf{x}}_i^{(i)}(t_{\lambda\mp})\right]\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\cdot\dot{\mathbf{x}}_i^{(j)}(t_{\lambda\mp})\right]>0,\\
&\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\cdot\dot{\mathbf{x}}_j^{(i)}(t_{\lambda\pm})\right]\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\cdot\dot{\mathbf{x}}_j^{(j)}(t_{\lambda\pm})\right]>0,\\
&\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\cdot\dot{\mathbf{x}}_i^{(i)}(t_{m_3\pm})\right]\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\cdot\dot{\mathbf{x}}_j^{(j)}(t_{m_3\mp})\right]=0
\end{aligned}\tag{5.48}$$

for $\mathbf{x}_{m_1}\in\overrightarrow{\partial\Omega}_{ij}$, $\mathbf{x}_{m_2}\in\overleftarrow{\partial\Omega}_{ij}$ *and* $\mathbf{x}_{m_3}\in\Gamma_{ij}$ *with* $\lambda\in\{m_1,m_2\}$.

(ii) *A nonpassable boundary* $(\partial\Omega_{ij}=\widetilde{\partial\Omega}_{ij}\cup\Gamma_{ij}\cup\widehat{\partial\Omega}_{ij})$ *exists iff*

$$\begin{aligned}
&\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\cdot\dot{\mathbf{x}}_i^{(i)}(t_{m_1-})\right]\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\cdot\dot{\mathbf{x}}_j^{(j)}(t_{m_1-})\right]<0,\\
&\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\cdot\dot{\mathbf{x}}_i^{(i)}(t_{m_2+})\right]\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\cdot\dot{\mathbf{x}}_j^{(j)}(t_{m_2+})\right]<0,\\
&\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\cdot\dot{\mathbf{x}}_i^{(i)}(t_{m_3\pm})\right]\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\cdot\dot{\mathbf{x}}_j^{(j)}(t_{m_3\pm})\right]=0
\end{aligned}\tag{5.49}$$

for $\mathbf{x}_{m_1}\in\widetilde{\partial\Omega}_{ij}$, $\mathbf{x}_{m_2}\in\widehat{\partial\Omega}_{ij}$ *and* $\mathbf{x}_{m_3}\in\Gamma_{ij}$.

(iii) *A mixed boundary of the first kind* ($\partial\Omega_{ij} = \overrightarrow{\partial\Omega}_{\alpha\beta} \cup \Gamma_{ij} \cup \widetilde{\partial\Omega}_{ij}$) *exists iff*

$$\begin{aligned}&\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}_{\alpha}(t_{m_1-})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}_{\alpha}(t_{m_1-})\big] > 0,\\ &\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(i)}_{i}(t_{m_2-})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(j)}_{j}(t_{m_2-})\big] < 0,\\ &\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(i)}_{i}(t_{m_3-})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \dot{\mathbf{x}}^{(j)}_{j}(t_{m_3-})\big] = 0\end{aligned} \tag{5.50}$$

for $\mathbf{x}_{m_1} \in \overrightarrow{\partial\Omega}_{\alpha\beta}$, $\mathbf{x}_{m_2} \in \widetilde{\partial\Omega}_{ij}$ *and* $\mathbf{x}_{m_3} \in \Gamma_{ij}$.

(iv) *A mixed boundary of the second kind* $\partial\Omega_{ij} = \overrightarrow{\partial\Omega}_{\alpha\beta} \cup \Gamma_{ij} \cup \widehat{\partial\Omega}_{ij}$ *iff*

$$\begin{aligned}&\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}_{\beta}(t_{m_1+})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}_{\beta}(t_{m_1+})\big] > 0,\\ &\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(i)}_{i}(t_{m_2+})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(j)}_{j}(t_{m_2+})\big] < 0,\\ &\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(i)}_{i}(t_{m_3+})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(j)}_{j}(t_{m_3+})\big] = 0\end{aligned} \tag{5.51}$$

for $\mathbf{x}_{m_1} \in \overrightarrow{\partial\Omega}_{\alpha\beta}$, $\mathbf{x}_{m_2} \in \widehat{\partial\Omega}_{ij}$ *and* $\mathbf{x}_{m_3} \in \Gamma_{ij}$.

PROOF. Using Theorem 5.9 and Lemma 5.1, this theorem is directly proved. □

THEOREM 5.12. *For a discontinuous dynamical system in Eq.* (5.21), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. *For arbitrarily small* $\varepsilon > 0$, *there are two time intervals* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$ *in the* δ*-layer of* $\partial\Omega_{ij}$. *Suppose* $\{\mathbf{x}^{(\alpha)}_{\alpha}(t_{m-}), \mathbf{x}^{(\beta)}_{\beta}(t_{m+})\} = \mathbf{x}_m$, $\{\mathbf{x}^{(\beta)}_{\alpha}(t_{m-}), \mathbf{x}^{(\alpha)}_{\beta}(t_{m+})\} = \mathbf{x}_m$ ($\alpha, \beta \in \{i, j\}$ *and* $\alpha \neq \beta$) *hold, the real and imaginary vector fields* $\{\mathbf{F}^{(\alpha)}_{\alpha}(t), \mathbf{F}^{(\alpha)}_{\beta}(t)\}$ *and* $\{\mathbf{F}^{(\beta)}_{\alpha}(t), \mathbf{F}^{(\beta)}_{\beta}(t)\}$ *are* $C^r_{[t_{m-\varepsilon}, t_m)}$*- and* $C^r_{(t_m, t_{m+\varepsilon}]}$*-continuous* ($r \geqslant 1$) *for time* t, *respectively,* $\|\mathrm{d}^{r+1}\mathbf{x}^{(\beta)}_{\alpha}/\mathrm{d}t^{r+1}\| < \infty$ ($\alpha, \beta \in \{i, j\}$) *and* $m \in \{m_1, m_2, m_3\}$.

(i) *A passable boundary* ($\partial\Omega_{ij} = \overrightarrow{\partial\Omega}_{ij} \cup \Gamma_{ij} \cup \overleftarrow{\partial\Omega}_{ij}$) *exists iff*

$$\begin{aligned}&\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}_{i}(t_{\lambda\mp})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}_{i}(t_{\lambda\mp})\big] > 0,\\ &\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}_{j}(t_{\lambda\pm})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}_{j}(t_{\lambda\pm})\big] > 0,\\ &\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}_{i}(t_{m_3\pm})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}_{j}(t_{m_3\mp})\big] = 0\end{aligned} \tag{5.52}$$

for $\mathbf{x}_{m_1} \in \overrightarrow{\partial\Omega}_{ij}$, $\mathbf{x}_{m_2} \in \overleftarrow{\partial\Omega}_{ij}$ *and* $\mathbf{x}_{m_3} \in \Gamma_{ij}$ *with* $\lambda \in \{m_1, m_2\}$.

(ii) *A nonpassable boundary* ($\partial\Omega_{ij} = \widetilde{\partial\Omega}_{ij} \cup \Gamma_{ij} \cup \widehat{\partial\Omega}_{ij}$) *exists iff*

$$\begin{aligned}&\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}_{i}(t_{m_1-})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}_{j}(t_{m_1-})\big] < 0,\\ &\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}_{i}(t_{m_2+})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}_{j}(t_{m_2+})\big] < 0,\\ &\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}_{i}(t_{m_3\pm})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}_{j}(t_{m_3\pm})\big] = 0\end{aligned} \tag{5.53}$$

for $\mathbf{x}_{m_1} \in \widetilde{\partial\Omega}_{ij}$, $\mathbf{x}_{m_2} \in \widehat{\partial\Omega}_{ij}$ *and* $\mathbf{x}_{m_3} \in \Gamma_{ij}$.

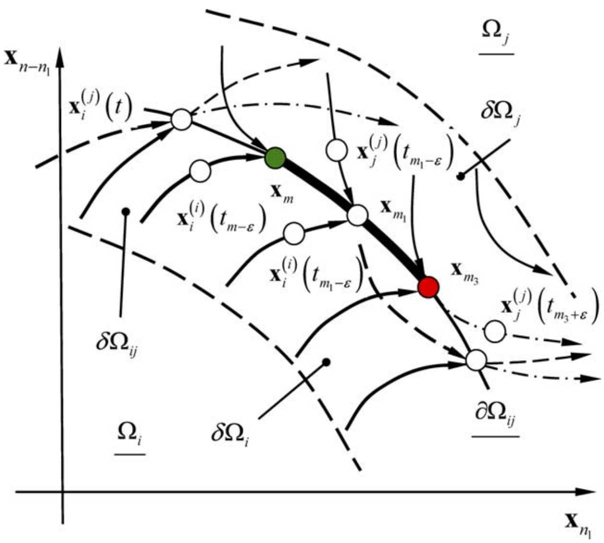


Figure 5.15. The onset, existence and disappearance of the sliding flows in the δ-sublayer of boundary $\partial\Omega_{ij} = \overrightarrow{\partial\Omega}_{ij} \cup \widetilde{\partial\Omega}_{ij} \cup \overrightarrow{\partial\Omega}_{ij}$.

(iii) *A mixed boundary of the first kind* ($\partial\Omega_{ij} = \overrightarrow{\partial\Omega}_{\alpha\beta} \cup \Gamma_{ij} \cup \widetilde{\partial\Omega}_{ij}$) *exists iff*

$$\begin{aligned}
&\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}_{\alpha}(t_{m_1-})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}_{\alpha}(t_{m_1-})\big] > 0,\\
&\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}_{i}(t_{m_2-})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}_{j}(t_{m_2-})\big] < 0,\\
&\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}_{i}(t_{m_3-})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}_{j}(t_{m_3-})\big] = 0
\end{aligned} \tag{5.54}$$

for $\mathbf{x}_{m_1} \in \overrightarrow{\partial\Omega}_{\alpha\beta}$, $\mathbf{x}_{m_2} \in \widetilde{\partial\Omega}_{ij}$ *and* $\mathbf{x}_{m_3} \in \Gamma_{ij}$.

(iv) *A mixed boundary of the second kind* $\partial\Omega_{ij} = \overrightarrow{\partial\Omega}_{\alpha\beta} \cup \Gamma_{ij} \cup \widehat{\partial\Omega}_{ij}$ *iff*

$$\begin{aligned}
&\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}_{\beta}(t_{p+})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}_{\beta}(t_{p+})\big] > 0,\\
&\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}_{i}(t_{q+})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}_{j}(t_{q+})\big] < 0,\\
&\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}_{i}(t_{n+})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}_{j}(t_{n+})\big] = 0
\end{aligned} \tag{5.55}$$

for $\mathbf{x}_n \in \Gamma_{ij}$, $\mathbf{x}_p \in \overrightarrow{\partial\Omega}_{\alpha\beta}$ *and* $\mathbf{x}_q \in \widehat{\partial\Omega}_{ij}$.

PROOF. Application of Definition 5.8 to Theorem 5.11 yields Eqs. (5.52)–(5.55). The theorem is proved. □

Using the definitions and theorems of the semi-passable, sink and source boundaries, the flow characteristics in the vicinity of the mixed boundary can be discussed and the corresponding theorem can be developed. Based on the foregoing theorem, the local nature of the flow near the mixed boundary can be analyzed through the investigation of the vector fields of the real and imaginary flows. The flow switching between the semi-passable flows or between nonpassable flows are presented. The onset, existence and disappearance of the sink and source flows can be presented through the real and imaginary flows. The flow switching at the gluing singular sets is described in the following definition. In addition, the concept of the normal vector field product in Chapter 3 can be also used in the real and imaginary flows. To demonstrate the concepts more intuitively, the normal vector field forms of the real and imaginary flows are still used in the further development.

DEFINITION 5.16. For a discontinuous dynamical system in Eq. (5.21), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. For arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$ in the δ-layer of $\partial\Omega_{ij}$. Suppose $\{\mathbf{x}_\alpha^{(\alpha)}(t_{m-}), \mathbf{x}_\beta^{(\beta)}(t_{m+})\} = \mathbf{x}_m$, $\{\mathbf{x}_\alpha^{(\beta)}(t_{m-}), \mathbf{x}_\beta^{(\alpha)}(t_{m+})\} = \mathbf{x}_m$ ($\alpha, \beta \in \{i, j\}$ and $\alpha \neq \beta$) hold, the real and imaginary flows $\{\mathbf{x}_\alpha^{(\alpha)}(t), \mathbf{x}_\beta^{(\alpha)}(t)\}$ and $\{\mathbf{x}_\alpha^{(\beta)}(t), \mathbf{x}_\beta^{(\beta)}(t)\}$ are $C^r_{[t_{m-\varepsilon}, t_m)}$- and $C^r_{(t_m, t_{m+\varepsilon}]}$-continuous ($r \geqslant 2$) for time t, respectively, $\|\mathrm{d}^r\mathbf{x}_\alpha^{(\beta)}/\mathrm{d}t^r\| < \infty$ ($\alpha, \beta \in \{i, j\}$) and $m \in \{m_1, m_2, m_3\}$.

(i) Two different semi-passable flows in the δ-layer of $\partial\Omega_{ij} = \overrightarrow{\partial\Omega}_{ij} \cup \Gamma_{ij} \cup \overleftarrow{\partial\Omega}_{ij}$ are switched on the boundary if

$$\begin{aligned}
&\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}_i^{(i)}(t_{m_1-})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}_i^{(i)}(t_{m_2+})\big] < 0,\\
&\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}_j^{(j)}(t_{m_1+})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}_j^{(j)}(t_{m_2-})\big] < 0,\\
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}_i^{(i)}(t_{m_3\pm}) = 0, \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}_j^{(j)}(t_{m_3\pm}) = 0
\end{aligned} \tag{5.56}$$

for $\mathbf{x}_{m_1} \in \overrightarrow{\partial\Omega}_{ij}$, $\mathbf{x}_{m_2} \in \overleftarrow{\partial\Omega}_{ij}$ and $\mathbf{x}_{m_3} \in \Gamma_{ij}$.

(ii) The source and sink flows in the δ-layer of $\partial\Omega_{ij} = \widetilde{\partial\Omega}_{ij} \cup \Gamma_{ij} \cup \widehat{\partial\Omega}_{ij}$ are switched on the boundary if

$$\begin{aligned}
&\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}_i^{(i)}(t_{m_1-})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}_i^{(i)}(t_{m_2+})\big] < 0,\\
&\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}_j^{(j)}(t_{m_1-})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}_j^{(j)}(t_{m_2+})\big] < 0,\\
&\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}_i^{(i)}(t_{m_3\pm})\big] = \big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}_j^{(j)}(t_{m_3\pm})\big] = 0
\end{aligned} \tag{5.57}$$

for $\mathbf{x}_{m_1} \in \widetilde{\partial\Omega}_{ij}$, $\mathbf{x}_{m_3} \in \widehat{\partial\Omega}_{ij}$ and $\mathbf{x}_{m_3} \in \Gamma_{ij}$.

(iii) The sliding flow in the δ-layer of $\partial\Omega_{ij} = \overrightarrow{\partial\Omega}_{\alpha\beta} \cup \Gamma_{ij} \cup \widetilde{\partial\Omega}_{ij}$ appears or disappears if

$$\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}_\alpha^{(\alpha)}(t_{m_1-})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}_\alpha^{(\alpha)}(t_{m_2-})\big] > 0,$$

$$\begin{aligned}
&\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\dot{\mathbf{x}}^{(\beta)}_{\alpha}(t_{m_1-})\right]\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\dot{\mathbf{x}}^{(\beta)}_{\beta}(t_{m_2-})\right]<0,\\
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\dot{\mathbf{x}}^{(\beta)}_{\beta}(t_{m_3-})=0,\quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\dot{\mathbf{x}}^{(\alpha)}_{\alpha}(t_{m_3-})\neq 0
\end{aligned}\tag{5.58}$$

for $\mathbf{x}_{m_1}\in\overrightarrow{\partial\Omega}_{\alpha\beta}$, $\mathbf{x}_{m_2}\in\widetilde{\partial\Omega}_{ij}$ and $\mathbf{x}_{m_3}\in\Gamma_{ij}$.

(iv) The source flow in the δ-layer of $\partial\Omega_{ij}=\overrightarrow{\partial\Omega}_{\alpha\beta}\cup\Gamma_{ij}\cup\widehat{\partial\Omega}_{ij}$ appears or disappears if

$$\begin{aligned}
&\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\dot{\mathbf{x}}^{(\alpha)}_{\beta}(t_{m_1+})\right]\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\dot{\mathbf{x}}^{(\alpha)}_{\alpha}(t_{m_2+})\right]<0,\\
&\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\dot{\mathbf{x}}^{(\beta)}_{\beta}(t_{m_1+})\right]\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\dot{\mathbf{x}}^{(\beta)}_{\beta}(t_{m_2+})\right]>0,\\
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\dot{\mathbf{x}}^{(\alpha)}_{\alpha}(t_{m_3+})=0,\quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\dot{\mathbf{x}}^{(\beta)}_{\beta}(t_{m_3+})\neq 0
\end{aligned}\tag{5.59}$$

for $\mathbf{x}_{m_1}\in\overrightarrow{\partial\Omega}_{\alpha\beta}$, $\mathbf{x}_{m_2}\in\widehat{\partial\Omega}_{ij}$ and $\mathbf{x}_{m_3}\in\Gamma_{ij}$.

From the above definition, the onset, existence and disappearance of the sliding flows on the boundary $\partial\Omega_{ij}=\overrightarrow{\partial\Omega}_{ij}\cup\widetilde{\partial\Omega}_{ij}\cup\overrightarrow{\partial\Omega}_{ij}$ in the δ-sublayer are intuitively illustrated in Fig. 5.15 through the real and imaginary flows. The solid and dash-dotted curves are used for $\mathbf{x}^{(\alpha)}_{\alpha}(t)$ $(\alpha\in\{i,j\})$ for $(t\in[t_{m-\varepsilon},t_{m-}))$ and $(t\in(t_{m+},t_{m+\varepsilon}])$ accordingly. The dashed curves depict the imaginary flows $\mathbf{x}^{(\beta)}_{\alpha}(t)$ $(\alpha\neq\beta\in\{i,j\})$ in the two domains. The dark and light curves represent the flows in Ω_i and Ω_j, respectively. The points $\mathbf{x}_m$ and $\mathbf{x}_n$ are the appearance and disappearance of the sliding motions. $\mathbf{x}_p\in\widetilde{\partial\Omega}_{ij}$ is all the points on the sliding boundary. Similarly, the onset, existence and disappearance of the source flows of the boundary $\partial\Omega_{ij}$ can be described. The switching between the two semi-passable boundaries and between the source and sink boundaries can be illustrated.

THEOREM 5.13. *For a discontinuous dynamical system in Eq. (5.21),* $\mathbf{x}(t_m)\equiv\mathbf{x}_m\in\partial\Omega_{ij}$ *for* t_m. *For arbitrarily small* $\varepsilon>0$, *there are two time intervals* $[t_{m-\varepsilon},t_m)$ *and* $(t_m,t_{m+\varepsilon}]$ *in the* δ*-layer of* $\partial\Omega_{ij}$. *Suppose* $\{\mathbf{x}^{(\alpha)}_{\alpha}(t_{m-}),\mathbf{x}^{(\beta)}_{\beta}(t_{m+})\}=\mathbf{x}_m$, $\{\mathbf{x}^{(\beta)}_{\alpha}(t_{m-}),\mathbf{x}^{(\alpha)}_{\beta}(t_{m+})\}=\mathbf{x}_m$ $(\alpha,\beta\in\{i,j\}$ *and* $\alpha\neq\beta)$ *hold, the real and imaginary vector fields* $\{\mathbf{F}^{(\alpha)}_{\alpha}(t),\mathbf{F}^{(\alpha)}_{\beta}(t)\}$ *and* $\{\mathbf{F}^{(\beta)}_{\alpha}(t),\mathbf{F}^{(\beta)}_{\beta}(t)\}$ *are* $C^r_{[t_{m-\varepsilon},t_m)}$*- and* $C^r_{(t_m,t_{m+\varepsilon}]}$*-continuous* $(r\geqslant 1)$ *for time* t, *respectively,* $\|\mathrm{d}^{r+1}\mathbf{x}^{(\beta)}_{\alpha}/\mathrm{d}t^{r+1}\|<\infty$ $(\alpha,\beta\in\{i,j\})$ *and* $m\in\{m_1,m_2,m_3\}$.

(i) *Two different semi-passable flows in the* δ*-layer of* $\partial\Omega_{ij}=\overrightarrow{\partial\Omega}_{ij}\cup\Gamma_{ij}\cup\overleftarrow{\partial\Omega}_{ij}$ *are switched on the boundary if*

$$\begin{aligned}
&\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\mathbf{F}^{(i)}_{i}(t_{m_1-})\right]\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\mathbf{F}^{(i)}_{i}(t_{m_2+})\right]<0,\\
&\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\mathbf{F}^{(j)}_{j}(t_{m_1+})\right]\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\mathbf{F}^{(j)}_{j}(t_{m_2-})\right]<0,
\end{aligned}\tag{5.60}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}_{i}(t_{m_3\pm}) = 0, \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}_{j}(t_{m_3\pm}) = 0$$

for $\mathbf{x}_{m_1} \in \overrightarrow{\partial\Omega}_{ij}$, $\mathbf{x}_{m_2} \in \overleftarrow{\partial\Omega}_{ij}$ *and* $\mathbf{x}_{m_3} \in \Gamma_{ij}$.

(ii) *The source and sink flows in the* δ*-layer of* $\partial\Omega_{ij} = \widetilde{\partial\Omega}_{ij} \cup \Gamma_{ij} \cup \widehat{\partial\Omega}_{ij}$ *are switched on the boundary if*

$$\begin{aligned} &\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}_{i}(t_{m_1-})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}_{i}(t_{m_2+})\big] < 0, \\ &\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}_{j}(t_{m_1-})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}_{j}(t_{m_2+})\big] < 0, \\ &\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}_{i}(t_{m_3\pm})\big] = \big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}_{j}(t_{m_3\pm})\big] = 0 \end{aligned} \tag{5.61}$$

for $\mathbf{x}_{m_1} \in \widetilde{\partial\Omega}_{ij}$, $\mathbf{x}_{m_3} \in \widehat{\partial\Omega}_{ij}$ *and* $\mathbf{x}_{m_3} \in \Gamma_{ij}$.

(iii) *The sliding flow in the* δ*-layer of* $\partial\Omega_{ij} = \overrightarrow{\partial\Omega}_{\alpha\beta} \cup \Gamma_{ij} \cup \widetilde{\partial\Omega}_{ij}$ *appears or disappears if*

$$\begin{aligned} &\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}_{\alpha}(t_{m_1-})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}_{\alpha}(t_{m_2-})\big] > 0, \\ &\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}_{\alpha}(t_{m_1-})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}_{\beta}(t_{m_2-})\big] < 0, \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}_{\beta}(t_{m_3-}) = 0, \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\mathbf{F}^{(\alpha)}_{\alpha}(t_{m_3-}) \neq 0 \end{aligned} \tag{5.62}$$

for $\mathbf{x}_{m_1} \in \overrightarrow{\partial\Omega}_{\alpha\beta}$, $\mathbf{x}_{m_2} \in \widetilde{\partial\Omega}_{ij}$ *and* $\mathbf{x}_{m_3} \in \Gamma_{ij}$.

(iv) *The source flow in the* δ*-layer of* $\partial\Omega_{ij} = \overrightarrow{\partial\Omega}_{\alpha\beta} \cup \Gamma_{ij} \cup \widehat{\partial\Omega}_{ij}$ *appears or disappears if*

$$\begin{aligned} &\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}_{\beta}(t_{m_1+})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}_{\alpha}(t_{m_2+})\big] < 0, \\ &\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot^{(\beta)}_{\beta}(t_{m_1+})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}_{\beta}(t_{m_2+})\big] > 0, \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}_{\alpha}(t_{m_3+}) = 0, \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}_{\beta}(t_{m_3+}) \neq 0 \end{aligned} \tag{5.63}$$

for $\mathbf{x}_{m_1} \in \overrightarrow{\partial\Omega}_{\alpha\beta}$, $\mathbf{x}_{m_2} \in \widehat{\partial\Omega}_{ij}$ *and* $\mathbf{x}_{m_3} \in \Gamma_{ij}$.

PROOF. Using Definitions 5.8, 5.10 and 5.16, the theorem is proved directly. □

The real and imaginary flows for nonsmooth dynamical systems are introduced. The theory for real flows in Chapters 2–4 can be applied to the imaginary flows. The δ-layer of the separation boundary is also introduced. The onset, existence and disappearance of the sink and source flows in the δ-layer are discussed, and the switching between the two semi-passable flows and the switching between the sink and source flows are investigated as well. Finally, the necessary and sufficient conditions for the onset, disappearance and switching are presented. These conditions can be very easily applied to nonsmooth dynamical systems in engineering, such as friction-induced vibrations, control systems with periodical excitations.

5.6. An example

Consider a piecewise linear dynamical system in two domains Ω_α ($\alpha \in \{1, 2\}$), given by

$$\ddot{x} + 2d_\alpha \dot{x} + c_\alpha x = b_\alpha + Q_0 \cos \Omega t \tag{5.64}$$

where b_α is constant rather than the control force in Eq. (4.80). The two domains are partitioned by the same boundary in Eq. (4.81). The domains partitioned by the boundary are sketched in Fig. 4.11. The domains and boundary definitions are given in Eqs. (4.82)–(4.84). The corresponding vector equations are given in Eqs. (4.85) and (4.86). The normal and tangential vectors of the boundary $\partial\Omega_{\alpha\beta}$ are

$$\mathbf{n}_{\partial\Omega_{\alpha\beta}} = (a_1, a_2)^{\mathrm{T}} \quad \text{and} \quad \mathbf{t}_{\partial\Omega_{\alpha\beta}} = (-a_2, a_1)^{\mathrm{T}}. \tag{5.65}$$

From Chapter 3, through the convexity, a dynamical equation of sliding motion on the boundary is determined by

$$\dot{\mathbf{x}}_{\alpha\beta} = \mathbf{A}\mathbf{x}_{\alpha\beta} \quad \text{on } \mathbf{x}_{\alpha\beta} \in \partial\Omega_{\alpha\beta} \tag{5.66}$$

where

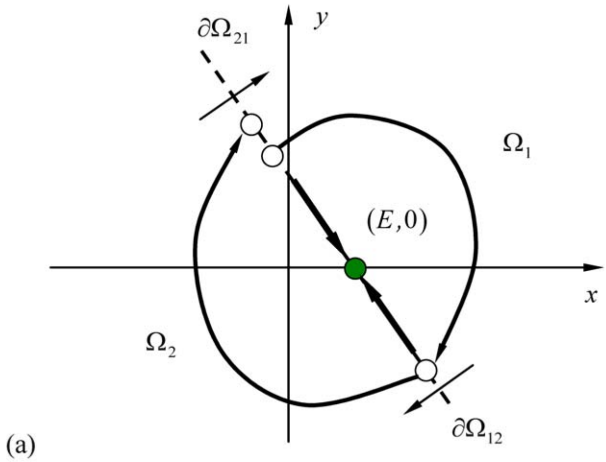
$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{a_1}{a_2} \end{bmatrix}. \tag{5.67}$$

From Eq. (5.66), the equilibrium of the sliding dynamical system on the boundary is given, i.e., $y^{\mathrm{e}}_{\alpha\beta} = 0$. From the boundary, the equilibrium is $(x^{\mathrm{e}}_{\alpha\beta}, y^{\mathrm{e}}_{\alpha\beta}) = (0, E)$ with $E = e/a_1$. At the equilibrium, the eigenvalue analysis requires

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 0-\lambda & 1 \\ 0 & -\frac{a_1}{a_2} - \lambda \end{vmatrix} = 0 \tag{5.68}$$

from which the two eigenvalues are $\lambda_1 = 0$, $\lambda_2 = -a_1/a_2$. If $a_1 a_2 > 0$, then the equilibrium is stable with the stable and center invariant eigenvector subspaces $E^{\mathrm{s}} = \mathrm{span}(1, -a_1/a_2)$, $E^{\mathrm{c}} = \mathrm{span}(1, 0)$. The unstable invariant eigenvector subspace is $E^{\mathrm{u}} = \varnothing$. However, if $a_1 a_2 < 0$, the equilibrium is unstable with invariant eigenvector subspaces $E^{\mathrm{u}} = \mathrm{span}(1, -a_1/a_2)$, $E^{\mathrm{c}} = \mathrm{span}(1, 0)$ and $E^{\mathrm{s}} = \varnothing$. In addition, the invariant subspaces should be on the boundary. The phase portraits in the equilibrium are illustrated in Fig. 5.16. In Figs. 5.16(a) and (b), the equilibrium possesses the parabolicity for two cases of $a_1 a_2$. However, the hyperbolicity of the equilibrium for the two cases of $a_1 a_2$ is also presented in Figs. 5.16(a) and (b). From Eq. (5.66), we have

$$x_{\alpha\beta} = E - \frac{a_2}{a_1} y^0_{\alpha\beta} \exp\left(-\frac{a_1}{a_2} t\right) \quad \text{and} \quad y_{\alpha\beta} = y^0_{\alpha\beta} \exp\left(-\frac{a_1}{a_2} t\right). \tag{5.69}$$



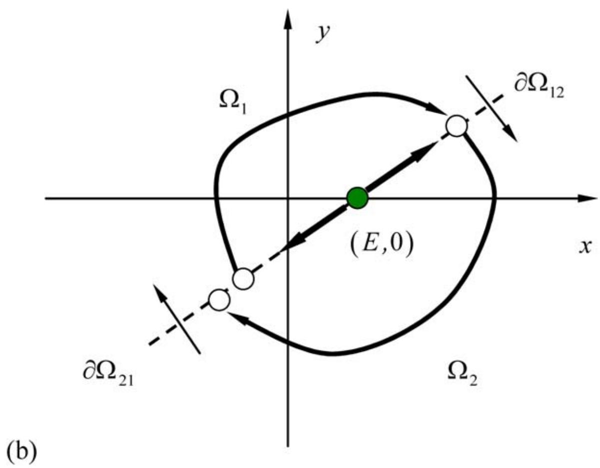


Figure 5.16. Phase portraits for flow parabolicity near equilibrium $(E, 0)$: (a) $a_1a_2 > 0$, and (b) $a_1a_2 < 0$.

Because of the sliding on the boundary, the initial condition should satisfy $a_1x^0_{\alpha\beta} + a_2y^0_{\alpha\beta} = e$. For the case $a_1a_2 > 0$, as $t \to \infty$, the sliding motion will approach the equilibrium whatever $y^0_{\alpha\beta}$ is. However, for $a_1a_2 < 0$, the equilibrium will be a source. The sliding flow for $y^0_{\alpha\beta} > 0$ and $y^0_{\alpha\beta} < 0$ will go to positive and negative infinities, respectively. For special cases of $a_1 = 0$ or $a_2 = 0$, the similar problems has been discussed in Chapter 2, and the detailed analysis can be referred to Luo and Chen (2006) and Luo and Gegg (2006a, 2006b). Consider

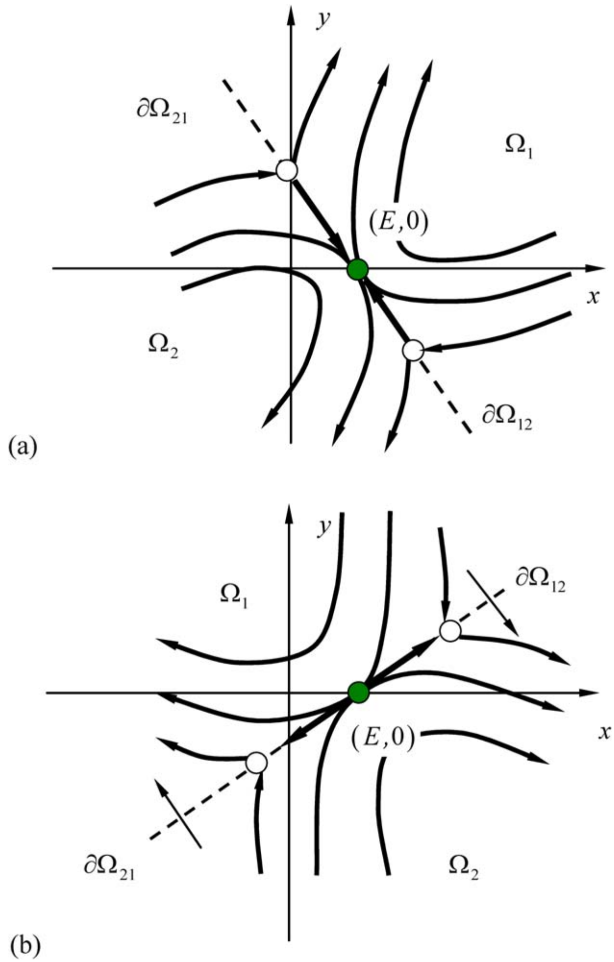


Figure 5.17. Phase portraits for flow hyperbolicity near equilibrium $(E, 0)$: (a) $a_1a_2 > 0$, and (b) $a_1a_2 < 0$.

the system parameters $a_1 = a_2 = 1, b_2 = -b_1 = 4, c_1 = 2, c_2 = 6, d_1 = d_2 = 0.01, Q_0 = 20$ for two subdomains. The simply periodic motion generated by the discontinuous system in Eq. (5.64) is presented for $\Omega = 1.5$ with the corresponding initial condition $(\Omega t_i, x_i, y_i) = (2.3232, -17.4104, 18.4104)$. The flow in phase plane shows a simple cycle with discontinuity through the dark solid curve in Fig. 5.18(a). The arrow direction is the flow direction. From the singular analysis of the separation boundary, the thin line presents the flow on the boundary. Such a discontinuity is proved through the normal vector fields $F_N^{(\alpha)} = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}$

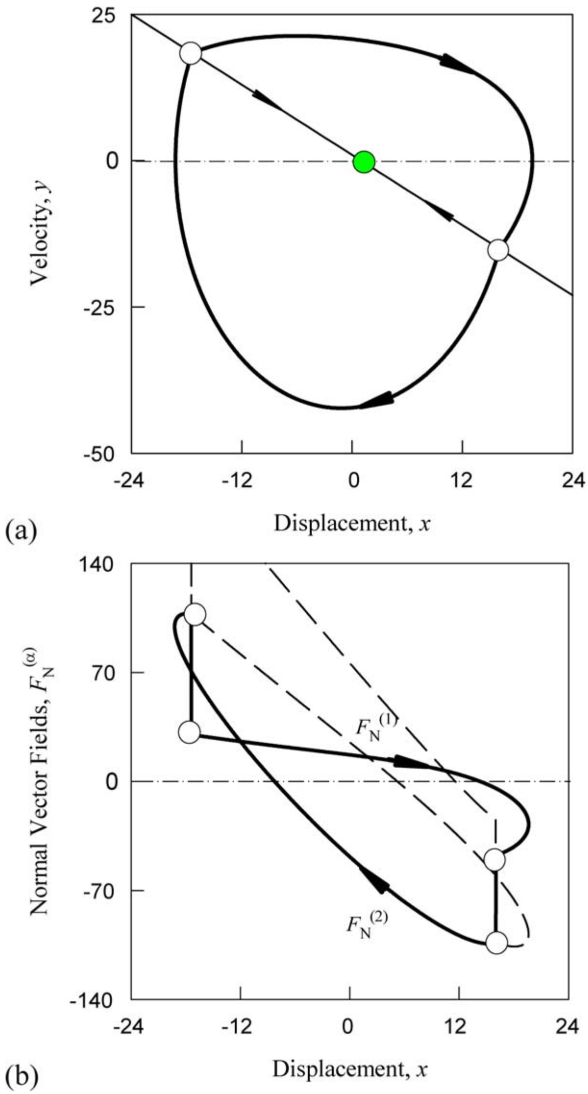
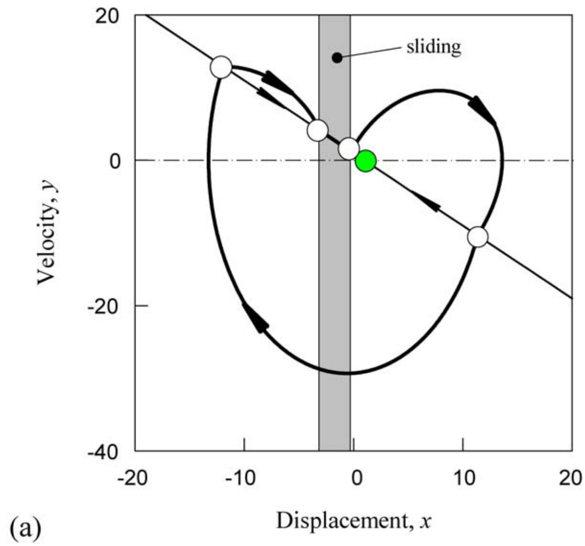


Figure 5.18. The parabolicity of the real flows in the vicinity of equilibrium on the boundary ($\Omega = 1.50$): (a) phase trajectory, and (b) the normal vector field to the boundary $\partial\Omega_{12}$. $(\Omega t_i, x_i, y_i) = (2.3232, -17.4104, 18.4104)$. ($a_1 = a_2 = 1$, $b_2 = -b_1 = 4$, $c_1 = 2$, $c_2 = 6$, $d_1 = d_2 = 0.01$, $Q_0 = 20$.)



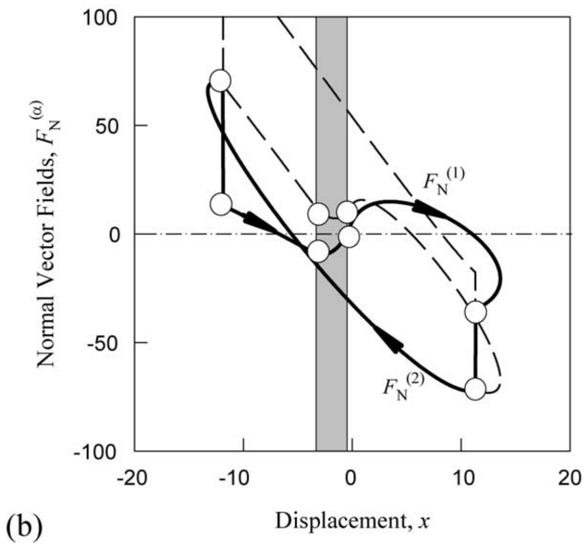


Figure 5.19. The flow with sliding on the boundary ($\Omega = 1.0$): (a) phase trajectory, and (b) the normal vector field to the separation boundary $\partial\Omega_{12}$. $(\Omega t_i, x_i, y_i) = (2.8762, -11.8917, 12.8917)$. ($a_1 = a_2 = 1, b_2 = -b_1 = 4, c_1 = 2, c_2 = 6, d_1 = d_2 = 0.01, Q_0 = 20.$)

($\alpha = 1, 2$), as shown in Fig. 5.18(b) with solid curves. The dashed curves are the imaginary normal vector fields corresponding to the trajectory of the real flow. The hollow symbols are for the flow switching. The filled circular symbol is the

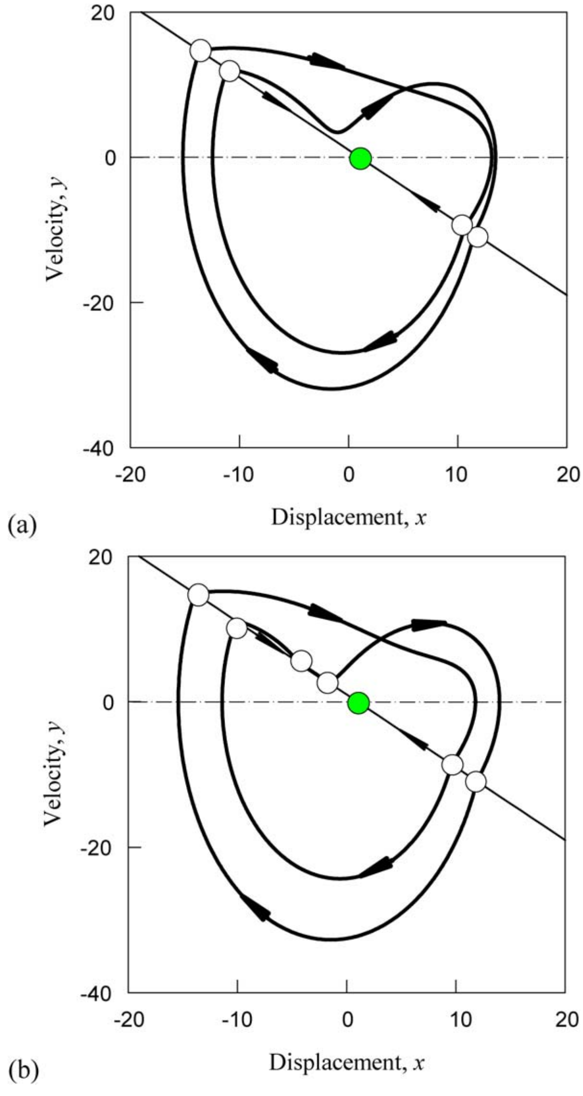


Figure 5.20. Period-doubling flows *with* and *without* sliding on the boundary $\partial\Omega_{12}$: (a) ($\Omega = 1.24$, $(\Omega t_i, x_i, y_i) = (2.5130, -11.1263, 12.1263)$), (b) ($\Omega = 1.2$, $(\Omega t_i, x_i, y_i) = (2.5820, -10.1267, 11.1267)$). ($a_1 = a_2 = 1$, $b_2 = -b_1 = 4$, $c_1 = 2$, $c_2 = 6$, $d_1 = d_2 = 0.01$, $Q_0 = 20$.)

equilibrium point of the separation boundary. The passability of the real flow to the discontinuous boundary is demonstrated. The phase trajectory in Fig. 5.18(a) shows the parabolicity of the equilibrium of the separation boundary, which is similar to the one in Fig. 5.16(a). To further verify the passability of a real flow with sliding consider the excitation $\Omega = 1.0$ with initial condition $((\Omega t_i, x_i, y_i) = (2.8762, -11.8917, 12.8917))$. The phase trajectory and vector field component are presented in Fig. 5.19. Except for passable portion of the boundary to the real flow, the sliding flow along the separation boundary is observed, which agrees with the aforementioned analysis in Eq. (5.69). The sliding motion disappearance satisfies the normal vector field conditions discussed before. In Fig. 5.20, the period-doubling, real flows without and with the sliding motion on the separation boundary are shown. The corresponding excitation frequencies are $\Omega = 1.24$ and $\Omega = 1.2$. It indicates the sliding bifurcation for the real flow sliding on the separation boundary will exist. The period-doubling real flows before and after the sliding bifurcation are generated by using the initial conditions (i.e., $(\Omega t_i, x_i, y_i) = (2.5130, -11.1263, 12.1263), (2.5820, -10.1267, 11.1267)$).

Chapter 6

Discontinuous Vector Fields with Flow Barriers

In previous chapters, the local singularities in discontinuous dynamical systems without flow barriers on the separation boundary were discussed and the corresponding sufficient and necessary conditions for such singularities were presented. The sliding flows on the boundary surface were investigated through the differential inclusion with the convexity, and the sliding fragmentation was also discussed. In this chapter, the flow passability on the separation boundary will be investigated in discontinuous dynamic systems with flow barriers. Because the flow barriers exist on the separation boundary, the singularities on such a separation boundary with flow barriers will be changed accordingly. Therefore, the necessary and sufficient conditions for the switching bifurcation and sliding fragmentation in discontinuous dynamical systems with flow barriers should be developed herein. A periodically forced friction model will be presented for illustration of a flow into the flow barrier with a force jumping.

6.1. Flow barriers

In Chapter 2, the nonpassable flow on the boundary $\partial\Omega_{ij}$ of the two domains Ω_i and Ω_j is dependent on the normal component product of the vector fields of two flows in the two domains. Assume a real flow $\mathbf{x}_\alpha^{(\alpha)}(t)$ $(\alpha \in \{i, j\})$ in the domain Ω_α for $t \in [t_{m-\varepsilon}, t_m)$ possesses $\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}_\alpha^{(\alpha)}(t) > 0$ in the vicinity of $\partial\Omega_{ij}$, and the flow $\mathbf{x}_\beta^{(\beta)}(t)$ in the domain Ω_β $(\alpha \neq \beta)$ possesses the condition $\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}_\beta^{(\beta)}(t) < 0$ in the vicinity of the boundary $\partial\Omega_{ij}$ at the same time interval. Thus, the sufficient and necessary conditions for the nonpassable flow on the boundary $\partial\Omega_{ij}$ are $[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}_\alpha^{(\alpha)}(t_{m\pm})][\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}_\beta^{(\beta)}(t_{m\pm})] < 0$ by Theorem 2.3. Furthermore, in Chapter 3, the sliding dynamics of the flow on the boundary $\partial\Omega_{ij}$ were discussed. The switching conditions from the passable and nonpassable motions were presented. The fragmentation of the sliding motion on the boundary was discussed. However, in many physical systems, the sliding motion is governed by $\dot{\mathbf{x}}_{\alpha\beta} = \mathbf{F}_{\alpha\beta}^{(0)}(\mathbf{x}_{\alpha\beta}, t)$ on the boundary $\partial\Omega_{ij}$. To keep the sliding motion on the boundary, there may be a normal flow barrier. To investigate the flow barrier on the separation boundary $\partial\Omega_{ij}$, as in Eqs. (5.21) and (5.22), consider a C^r-continuous system ($r \geqslant 1$) on Ω_α ($\alpha = i, j$) in a form of

$$\dot{\mathbf{x}}_\alpha^{(\gamma)} = \mathbf{F}^{(\gamma)}\big(\mathbf{x}_\alpha^{(\gamma)}, t, \boldsymbol{\mu}_\gamma\big) \in \Re^n, \quad \mathbf{x}_\alpha^{(\gamma)} = \big(x_{\alpha 1}^{(\gamma)}, x_{\alpha 2}^{(\gamma)}, \ldots, x_{\alpha n}^{(\gamma)}\big)^{\mathrm{T}} \in \Omega_\alpha \tag{6.1}$$

($\gamma = \alpha, \beta$) with the initial condition

$$\mathbf{x}_\alpha^{(\gamma)}(t_0) = \boldsymbol{\Phi}^{(\gamma)}\big(\mathbf{x}_\alpha^{(\gamma)}(t_0), t_0, \boldsymbol{\mu}_\gamma\big). \tag{6.2}$$

For $\gamma = \alpha$, the flow determined by Eqs. (6.1) and (6.2) is the real flow on the domain Ω_α. However, for $\gamma = \beta$, the flow determined by Eqs. (6.1) and (6.2) is the imaginary flow of the β-vector field on the domain Ω_α. Consider a vector field

$$\mathbf{F}^{(\gamma)}\big(\mathbf{x}_\alpha^{(\gamma)}, t, \boldsymbol{\mu}_\gamma\big) \equiv \mathbf{f}^{(\gamma)}\big(\mathbf{x}_\alpha^{(\gamma)}, t, \boldsymbol{\mu}_\gamma\big) + \mathbf{g}\big(\mathbf{x}_\alpha^{(\gamma)}, t\big). \tag{6.3}$$

Consider a dynamical system on the boundary $\Omega_{\alpha\beta}$ ($\beta \neq \alpha$ and $\alpha, \beta \in \{i, j\}$) in the following form:

$$\dot{\mathbf{x}}_{\alpha\beta} = \mathbf{F}_{\alpha\beta}(\mathbf{x}_{\alpha\beta}, t), \quad \mathbf{x}_{\alpha\beta} = (x_{1\alpha\beta}, x_{2\alpha\beta}, \ldots, x_{n\alpha\beta})^{\mathrm{T}} \in \partial\Omega_{\alpha\beta}. \tag{6.4}$$

For a sliding flow on the boundary, we have

$$\mathbf{F}_{\alpha\beta}(\mathbf{x}_{\alpha\beta}, t) = \mathbf{F}_{\alpha\beta}^{(0)}(\mathbf{x}_{\alpha\beta}, t, \boldsymbol{\mu}_{\alpha\beta}) \quad \text{on } \widetilde{\partial\Omega}_{\alpha\beta}. \tag{6.5}$$

If only the sliding flow on the boundary exists, the normal component of the vector field $\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}_{\alpha\beta}(\mathbf{x}_{\alpha\beta}, t)$ will lie in the *lower* and *upper* flow barriers. Such flow barriers will be defined as follows.

DEFINITION 6.1. For a discontinuous dynamical system with subsystems in Eq. (6.1), there is a nonpassable flow on the boundary $\partial\Omega_{ij}$. For a flow $\mathbf{x}_{\alpha\beta}(t) \in \partial\Omega_{ij}$ ($\alpha, \beta \in \{i, j\}$ and $\alpha \neq \beta$), there are two vector fields $\mathbf{F}_{\alpha\beta}^{(\alpha)}(\mathbf{x}_{\alpha\beta}, t)$ and $\mathbf{F}_{\alpha\beta}^{(\beta)}(\mathbf{x}_{\alpha\beta}, t)$ on the boundary $\partial\Omega_{ij}$, and $\mathbf{n}_{\partial\Omega_{\alpha\beta}}^{\mathrm{T}} \cdot [\mathbf{F}_{\alpha\beta}^{(\alpha)}(\mathbf{x}_{\alpha\beta}, t) - \mathbf{F}_{\alpha\beta}^{(\beta)}(\mathbf{x}_{\alpha\beta}, t)] < 0$. If

$$\begin{aligned}
&\mathbf{n}_{\partial\Omega_{\alpha\beta}}^{\mathrm{T}} \cdot \mathbf{f}^{(\alpha)}(\mathbf{x}_{\alpha\beta}, t, \boldsymbol{\mu}_\alpha) \in \big[\mathbf{n}_{\partial\Omega_{\alpha\beta}}^{\mathrm{T}} \cdot \mathbf{F}_{\alpha\beta}^{(\alpha)}(\mathbf{x}_{\alpha\beta}, t), \mathbf{n}_{\partial\Omega_{\alpha\beta}}^{\mathrm{T}} \cdot \mathbf{F}_{\alpha\beta}^{(\beta)}(\mathbf{x}_{\alpha\beta}, t)\big), \\
&\mathbf{n}_{\partial\Omega_{\alpha\beta}}^{\mathrm{T}} \cdot \mathbf{f}^{(\beta)}(\mathbf{x}_{\alpha\beta}, t, \boldsymbol{\mu}_\beta) \in \big(\mathbf{n}_{\partial\Omega_{\alpha\beta}}^{\mathrm{T}} \cdot \mathbf{F}_{\alpha\beta}^{(\alpha)}(\mathbf{x}_{\alpha\beta}, t), \mathbf{n}_{\partial\Omega_{\alpha\beta}}^{\mathrm{T}} \cdot \mathbf{F}_{\alpha\beta}^{(\beta)}(\mathbf{x}_{\alpha\beta}, t)\big]
\end{aligned} \tag{6.6}$$

then the vector fields $\mathbf{F}_{\alpha\beta}^{(\alpha)}(\mathbf{x}_{\alpha\beta}, t)$ and $\mathbf{F}_{\alpha\beta}^{(\beta)}(\mathbf{x}_{\alpha\beta}, t)$ are termed *lower* and *upper* flow barriers on the boundary $\partial\Omega_{ij}$.

From Definition 6.1, the lower and upper flow barriers on the boundary are sketched in Fig. 6.1 from the normal components of the vector fields on the boundary. The flow barriers ($\mathbf{F}_{\alpha\beta}^{(\alpha)}$ and $\mathbf{F}_{\alpha\beta}^{(\beta)}$) and the domain vector fields ($\mathbf{f}^{(\alpha)}$ and $\mathbf{f}^{(\beta)}$)

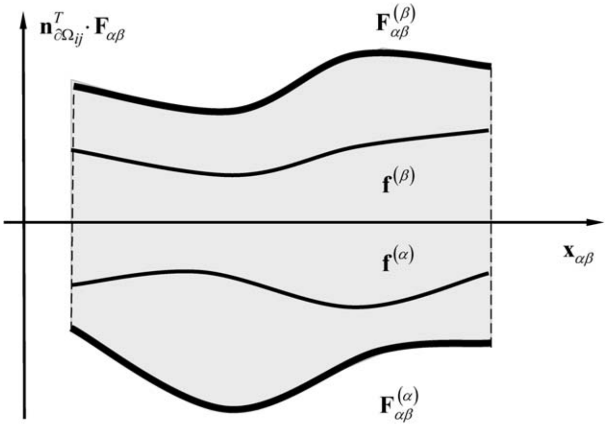


Figure 6.1. Flow barriers on the boundary $\partial\Omega_{\alpha\beta}$. The flow barriers ($\mathbf{F}^{(\alpha)}_{\alpha\beta}$ and $\mathbf{F}^{(\beta)}_{\alpha\beta}$) and the domain vector fields ($\mathbf{f}^{(\alpha)}$ and $\mathbf{f}^{(\beta)}$) on the boundary are depicted by dark and thin curves, respectively.

on the boundary are depicted by dark and thin curves, respectively. The flow barriers form a gray-area to govern the flow vector fields on the boundary. If the following equations hold,

$$\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{f}^{(\alpha)}(\mathbf{x}_{\alpha\beta}, t) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}_{\alpha\beta}(\mathbf{x}_{\alpha\beta}, t) \quad \text{and} \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{f}^{(\beta)}(\mathbf{x}_{\alpha\beta}, t) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}_{\alpha\beta}(\mathbf{x}_{\alpha\beta}, t), \end{aligned} \tag{6.7}$$

then the flow barriers disappear. For this case, the passable and nonpassable motions and the corresponding singularities were discussed in the previous chapters. The sufficient and necessary conditions for the passable motion are

$$\begin{aligned} &\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(\mathbf{x}_m, t_{m-})\big] < 0 \quad \text{and} \quad \big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}(\mathbf{x}_m, t_{m+})\big] < 0 \\ &\quad \text{for } \overrightarrow{\partial\Omega}_{\alpha\beta} \text{ convex to } \Omega_\alpha, \end{aligned} \tag{6.8a}$$

$$\begin{aligned} &\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}(\mathbf{x}_m, t_{m-})\big] > 0 \quad \text{and} \quad \big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(\mathbf{x}_m, t_{m+})\big] > 0 \\ &\quad \text{for } \overrightarrow{\partial\Omega}_{\alpha\beta} \text{ convex to } \Omega_\beta. \end{aligned} \tag{6.8b}$$

If $\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}_{\alpha\beta}(\mathbf{x}_{\alpha\beta}, t) \to -\infty$ and $\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}_{\alpha\beta}(\mathbf{x}_{\alpha\beta}, t)$ is finite, then the flow barrier is the *lower semi-permanent barrier*. The vector fields on both sides of $\partial\Omega_{ij}$ lie in the following interval, i.e.,

$$\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{f}^{(\alpha)}(\mathbf{x}_{\alpha\beta}, t) \in \big(-\infty, \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}_{\alpha\beta}(\mathbf{x}_{\alpha\beta}, t)\big), \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{f}^{(\beta)}(\mathbf{x}_{\alpha\beta}, t) \in \big(-\infty, \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}_{\alpha\beta}(\mathbf{x}_{\alpha\beta}, t)\big]. \end{aligned} \tag{6.9}$$

If the aforementioned flow barrier exists, the flow on the boundary only exits from *the upper flow barrier*. The flow on the boundary cannot exit from the lower permanent flow barrier. Similarly, if $\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}_{\alpha\beta}(\mathbf{x}_{\alpha\beta}, t) \to \infty$ and $\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}_{\alpha\beta}(\mathbf{x}_{\alpha\beta}, t)$ is finite, then the flow barrier is the *upper semi-permanent barrier*. The vector fields on both sides of $\partial\Omega_{ij}$ lie in the following interval:

$$\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{f}^{(\alpha)}(\mathbf{x}_{\alpha\beta}, t) \in \big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}_{\alpha\beta}(\mathbf{x}_{\alpha\beta}, t), \infty\big), \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{f}^{(\beta)}(\mathbf{x}_{\alpha\beta}, t) \in \big(\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}_{\alpha\beta}(\mathbf{x}_{\alpha\beta}, t), \infty\big). \end{aligned} \tag{6.10}$$

From the foregoing conditions, the flow on the boundary only exits from the lower flow barrier. If two barrier vector fields are infinite (i.e., $\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}_{\alpha\beta}(\mathbf{x}_{\alpha\beta}, t) \to -\infty$ and $\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}_{\alpha\beta}(\mathbf{x}_{\alpha\beta}, t) \to \infty$), then the flow barrier on the boundary is fully permanent. We have

$$\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{f}^{(\alpha)}(\mathbf{x}_{\alpha\beta}, t) \in (-\infty, \infty), \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{f}^{(\beta)}(\mathbf{x}_{\alpha\beta}, t) \in (-\infty, \infty). \end{aligned} \tag{6.11}$$

On the flow on the boundary, the flow cannot exit from boundary. Since the flow barriers exist, the following concept is introduced.

DEFINITION 6.2. For a discontinuous dynamical system with subsystems in Eq. (6.1), there is a nonpassable flow on the boundary $\partial\Omega_{ij}$ with two flow barriers $\mathbf{F}^{(i)}_{ij}(\mathbf{x}_{ij}, t)$ and $\mathbf{F}^{(j)}_{ij}(\mathbf{x}_{ij}, t)$. The *input and output vector fields* of the α-domain Ω_α on the boundary $\partial\Omega_{\alpha\beta}$ for $\alpha, \beta \in \{i, j\}$ and $\alpha \neq \beta$ are defined as

$$\begin{aligned} &\mathbf{F}^{(\alpha)}_{\mathrm{in}}(t_{m\pm}) \equiv \mathbf{f}^{(\alpha)}_{\alpha\beta}(\mathbf{x}_m, t_{m\pm}, \boldsymbol{\mu}_\alpha) + \mathbf{g}(\mathbf{x}_m, t_{m\pm}), \\ &\mathbf{F}^{(\alpha)}_{\mathrm{b}}(t_{m\pm}) \equiv \mathbf{F}^{(\alpha)}_{\alpha\beta}(\mathbf{x}_m, t_{m\pm}, \boldsymbol{\mu}_\alpha) + \mathbf{g}(\mathbf{x}_m, t_{m\pm}). \end{aligned} \tag{6.12}$$

For passable flows on the boundary, the flow is independent of the flow barriers. Theorem 2.2 can be applied and the corresponding vector field will be replaced by $\mathbf{F}^{(\alpha)}_{\mathrm{in}}(t_{m\pm})$,

$$\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}_{\mathrm{in}}(t_{m-})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}_{\mathrm{in}}(t_{m+})\big] > 0. \tag{6.13}$$

The necessary and sufficient conditions for the onset of the sink and source flows on the boundary are given by Theorems 2.4 and 2.6, respectively. Namely, when the flow arrives to the boundary $\partial\Omega_{\alpha\beta}$, once the following conditions hold,

$$\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}_{\mathrm{in}}(t_{m-})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}_{\mathrm{in}}(t_{m-})\big] < 0, \tag{6.14}$$

the sink flow appears on such a boundary. Once the sink motion on the boundary is formed, the flow must overcome the flow barrier. Without the flow barrier, the conditions in Eq. (6.14) will govern the sink flow until appearance, and the corresponding conditions for vanishing of the sink flow on the boundary requires

$$\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}_{\mathrm{in}}(t_{m-})\right]\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}_{\mathrm{in}}(t_{m-})\right] = 0. \tag{6.15}$$

Similarly, the conditions for the existence of the source flow on the boundary can be determined by

$$\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}_{\mathrm{in}}(t_{m+})\right]\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}_{\mathrm{in}}(t_{m+})\right] < 0, \tag{6.16}$$

and the onset condition for the source motion on the boundary is

$$\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}_{\mathrm{in}}(t_{m+})\right]\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}_{\mathrm{in}}(t_{m+})\right] = 0. \tag{6.17}$$

However, for a discontinuous dynamical system with flow barriers on the boundary, when the flow arrives to the boundary, if conditions in Eqs. (6.14) or (6.16) hold, a sink or source flow will be formed on the boundary. The existence of the sink and source flows will be governed by the flow barriers, and the corresponding conditions, respectively, are

$$\begin{aligned}&\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}_{\mathrm{b}}(t_{m-})\right]\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}_{\mathrm{b}}(t_{m-})\right] < 0 \quad \text{and}\\ &\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}_{\mathrm{b}}(t_{m+})\right]\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}_{\mathrm{b}}(t_{m+})\right] < 0.\end{aligned} \tag{6.18}$$

Once one of the two normal vector field products becomes zero, the sink or source flow will disappear and become a passable flow on the boundary. Namely, the conditions for vanishing of the sink or source flow on the boundary are

$$\begin{aligned}&\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}_{\mathrm{b}}(t_{m-})\right]\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}_{\mathrm{b}}(t_{m-})\right] = 0 \quad \text{or}\\ &\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}_{\mathrm{b}}(t_{m+})\right]\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}_{\mathrm{b}}(t_{m+})\right] = 0\end{aligned} \tag{6.19}$$

for a nonsmooth dynamic system with flow barriers. After the sink or source flow vanishes from the boundary with flow barriers, the flow on the boundary will be controlled by Eq. (6.1). For instance, if $\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}_{\mathrm{b}}(t_{m-}) = 0$ for the sink flow, the flow on the boundary will be controlled by Eq. (6.1) in the domain Ω_β. However, $\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}_{\mathrm{in}}(t_{m+})$ will be an initial normal component of the vector flow for the motion in the domain Ω_β.

The concept of Definition 6.1 can be extended, and the flow barriers can exist on each side of the boundary. Therefore, the α-flow barriers on the boundary of the domain Ω_α can be defined. Furthermore, the domain vector fields ($\mathbf{f}^{(\alpha)}$ and $\mathbf{f}^{(\beta)}$) on the boundary may not satisfy Eq. (6.6). Once the nonpassable flow on

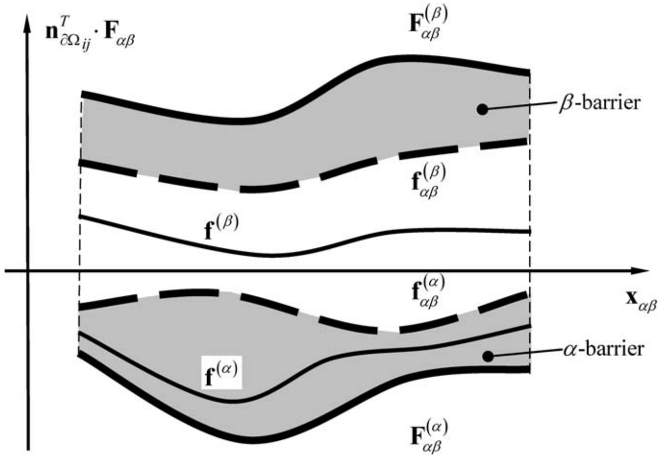


Figure 6.2. The α- and β-flow barriers on the boundary $\partial\Omega_{\alpha\beta}$. The output flow barriers ($\mathbf{F}^{(\alpha)}_{\alpha\beta}$ and $\mathbf{F}^{(\beta)}_{\alpha\beta}$), input flow barriers ($\mathbf{f}^{(\alpha)}_{\alpha\beta}$ and $\mathbf{f}^{(\beta)}_{\alpha\beta}$), and the domain vector fields ($\mathbf{f}^{(\alpha)}$ and $\mathbf{f}^{(\beta)}$) on the boundary are depicted by dark, dashed, and thin curves, respectively.

the boundary appears, the vector field in domain switches to the corresponding input flow barrier on the boundary. This idea is sketched in Fig. 6.2. The output flow barriers ($\mathbf{F}^{(\alpha)}_{\alpha\beta}$ and $\mathbf{F}^{(\beta)}_{\alpha\beta}$), input flow barriers ($\mathbf{f}^{(\alpha)}_{\alpha\beta}$ and $\mathbf{f}^{(\beta)}_{\alpha\beta}$), and the domain vector fields ($\mathbf{f}^{(\alpha)}$ and $\mathbf{f}^{(\beta)}$) on the boundary are depicted by dark, dashed, and thin curves, respectively. The α- and β-barriers are depicted through the shaded areas. $\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{f}^{(\alpha)} \in (\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{f}^{(\alpha)}_{\alpha\beta}, \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}_{\alpha\beta}]$ on the boundary, but $\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{f}^{(\beta)} \notin (\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{f}^{(\beta)}_{\alpha\beta}, \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}_{\alpha\beta}]$. Therefore, the flow barriers can be defined arbitrarily. In addition, on the boundary, for some $\mathbf{x}_m \in \partial\Omega_{\alpha\beta}$, the domain vector field may be in the flow barrier or not.

DEFINITION 6.3. For a discontinuous dynamical system with subsystems in Eq. (6.1), there is a nonpassable flow on the boundary $\partial\Omega_{ij}$. For a flow $\mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m ($\alpha, \beta \in \{i, j\}$ and $\alpha \neq \beta$), there are two vector fields $\mathbf{f}^{(\alpha)}_{\alpha\beta}(\mathbf{x}_m, t_{m\pm})$ and $\mathbf{F}^{(\alpha)}_{\alpha\beta}(\mathbf{x}_m, t_{m\pm})$ on the α-side of the boundary $\partial\Omega_{ij}$. For the input flow from the domain Ω_α into the boundary with following condition,

$$\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(i)}_{i}(t_{m\pm})\right]\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}_{j}(t_{m\pm})\right] < 0, \tag{6.20}$$

if the corresponding vector field is switched as

$$\mathbf{f}^{(\alpha)}(\mathbf{x}_m, t_{m\pm}) \to \mathbf{f}^{(\alpha)}_{\alpha\beta}(\mathbf{x}_m, t_{m\pm}), \tag{6.21}$$

the vector field $\mathbf{f}_{\alpha\beta}^{(\alpha)}(\mathbf{x}_m, t_{m\pm})$ is termed *the input flow barrier on the* α*-side* of the boundary $\partial\Omega_{ij}$. For the output flow on the boundary into the domain Ω_α with the following condition,

$$\mathbf{n}_{\partial\Omega_{\alpha\beta}}^{\mathrm{T}} \cdot \mathbf{F}_{\mathrm{b}}^{(\alpha)}(t_{m\pm}) = 0 \quad \text{and} \quad \mathbf{n}_{\partial\Omega_{\alpha\beta}}^{\mathrm{T}} \cdot \mathbf{F}_{\mathrm{b}}^{(\beta)}(t_{m\pm}) \neq 0, \tag{6.22}$$

if the corresponding vector field is switched as

$$\mathbf{F}_{\alpha\beta}^{(\alpha)}(\mathbf{x}_m, t_{m\pm}) \to \mathbf{f}^{(\alpha)}(\mathbf{x}_m, t_{m\pm}), \tag{6.23}$$

the vector field $\mathbf{F}_{\alpha\beta}^{(\alpha)}(\mathbf{x}_m, t_{m\pm})$ is termed *the output flow barrier on the* α*-side* of the boundary $\partial\Omega_{ij}$.

6.2. Switching bifurcations

From the discussions in Section 6.1, the switching bifurcation conditions can be summarized in the following theorems, which can be obtained when the vector field $\mathbf{F}_{\mathrm{in}}^{(\alpha)}(t_{m\pm})$ on the boundary is used. The necessary and sufficient conditions for the switching bifurcation from the passable to the nonpassable boundaries are presented as follows.

THEOREM 6.1. *For a discontinuous dynamical system with subsystems in Eq. (6.1), a nonpassable flow on the boundary* $\partial\Omega_{ij}$ *is governed by two flow barriers* $\mathbf{F}_{ij}^{(i)}(\mathbf{x}_{ij}, t)$ *and* $\mathbf{F}_{ij}^{(j)}(\mathbf{x}_{ij}, t)$. *For a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ *for* t_m, *there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$*) for an arbitrarily small* $\varepsilon > 0$, *and* $\mathbf{x}_i^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}_j^{(j)}(t_{m\pm})$. *The real flows* $\mathbf{x}_i^{(i)}(t)$ *and* $\mathbf{x}_j^{(j)}(t)$ *are* $C^r_{[t_{m-\varepsilon}, t_m)}$*- and* $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 1)$ *for time* t, *respectively. The sliding bifurcation of the flow* $\mathbf{x}_i^{(i)}(t) \cup \mathbf{x}_j^{(j)}(t)$ *on the boundary* $\overrightarrow{\partial\Omega}_{ij}$ *exists iff*

$$\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}_{\mathrm{in}}^{(j)}(t_{m\pm}) = 0 \quad \textit{and} \quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}_{\mathrm{in}}^{(i)}(t_{m-}) \neq 0; \tag{6.24}$$

$$\left.\begin{aligned} &\textit{either} \quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}_{\mathrm{in}}^{(i)}(t_{m-}) > 0 \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ &\textit{or} \quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}_{\mathrm{in}}^{(i)}(t_{m-}) < 0 \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i; \end{aligned}\right. \tag{6.25}$$

$$\left.\begin{aligned} &\textit{either} \quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \left[\mathrm{D}\mathbf{F}_{\mathrm{in}}^{(j)}(t_{m\pm}) - \mathrm{D}\mathbf{F}_{ij}^{(0)}(t_{m\pm})\right] > 0 \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ &\textit{or} \quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \left[\mathrm{D}\mathbf{F}_{\mathrm{in}}^{(j)}(t_{m\pm}) - \mathrm{D}\mathbf{F}_{ij}^{(0)}(t_{m\pm})\right] < 0 \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{aligned}\right. \tag{6.26}$$

PROOF. Consider the input flow into the flow barriers without the jumping, and then input vector fields in the domains Ω_α and Ω_β are

$$\mathbf{F}_{\text{in}}^{(\alpha)}\big(\mathbf{x}_\alpha^{(\alpha)}, t, \boldsymbol{\mu}_\alpha\big) \equiv \mathbf{f}^{(\alpha)}\big(\mathbf{x}_\alpha^{(\alpha)}, t, \boldsymbol{\mu}_\alpha\big) + \mathbf{g}(\mathbf{x}, t),$$
$$\mathbf{F}_{\text{in}}^{(\beta)}\big(\mathbf{x}_\beta^{(\beta)}, t, \boldsymbol{\mu}_\beta\big) \equiv \mathbf{f}^{(\beta)}\big(\mathbf{x}_\beta^{(\beta)}, t, \boldsymbol{\mu}_\beta\big) + \mathbf{g}(\mathbf{x}, t)$$

identical to the vector field in Eq. (6.1). The conditions in Eqs. (6.24)–(6.26) can be obtained from the necessary and sufficient conditions in Theorem 3.4 for the switching bifurcation. If the input, vector fields are different from Eq. (6.1), from the first equation of Eq. (6.12) in Definition 6.2, two *imaginary* input vector fields are constructed through the input vector fields $\mathbf{f}_{\alpha\beta}^{(\alpha)}(\mathbf{x}_{\alpha\beta}, t_m, \boldsymbol{\mu}_\alpha)$ and $\mathbf{f}_{\alpha\beta}^{(\beta)}(\mathbf{x}_{\alpha\beta}, t_m, \boldsymbol{\mu}_\beta)$ of the flow barriers as

$$\mathbf{F}_{\text{in}}^{(\alpha)}\big(\mathbf{x}_\alpha^{(\alpha)}, t, \boldsymbol{\mu}_\alpha\big) \equiv \mathbf{f}_{\alpha\beta}^{(\alpha)}\big(\mathbf{x}_\alpha^{(\alpha)}, t, \boldsymbol{\mu}_\alpha\big) + \mathbf{g}(\mathbf{x}, t),$$
$$\mathbf{F}_{\text{in}}^{(\beta)}\big(\mathbf{x}_\beta^{(\beta)}, t, \boldsymbol{\mu}_\beta\big) \equiv \mathbf{f}_{\alpha\beta}^{(\beta)}\big(\mathbf{x}_\beta^{(\beta)}, t, \boldsymbol{\mu}_\beta\big) + \mathbf{g}(\mathbf{x}, t).$$

For a point $\mathbf{x}_m \in \partial\Omega_{\alpha\beta}$ at time t_m, as time $t \to t_{m-}$, the flow $\mathbf{x}_\alpha^{(\alpha)} \to \mathbf{x}_m$ and the corresponding vector field $\mathbf{F}_{\text{in}}^{(\alpha)}(\mathbf{x}_\alpha^{(\alpha)}, t, \boldsymbol{\mu}_\alpha) \to \mathbf{F}_{\text{in}}^{(\alpha)}(\mathbf{x}_m, t_{m-}, \boldsymbol{\mu}_\alpha)$. From Definition 6.2, $\mathbf{F}_{\text{in}}^{(\alpha)}(\mathbf{x}_m, t_{m-}, \boldsymbol{\mu}_\alpha) = \mathbf{F}_{\text{in}}^{(\alpha)}(t_{m-})$. Such relations are suitable for the imaginary vector field in the domain Ω_β. Further, a new imaginary discontinuous dynamic system is given by

$$\dot{\mathbf{x}}_\alpha^{(\alpha)} = \mathbf{F}_{\text{in}}^{(\alpha)}\big(\mathbf{x}_\alpha^{(\alpha)}, t, \boldsymbol{\mu}_\alpha\big), \quad \alpha \in \{i, j\}.$$

Using Definition 3.7 for the sliding fragmentation, following the proof procedure of Theorems 2.10–2.11, Eqs. (6.24)–(6.26) can be obtained. □

THEOREM 6.2. *For a discontinuous dynamical system with subsystems in Eq.* (6.1)*, a nonpassable flow on the boundary $\partial\Omega_{ij}$ is governed by two flow barriers $\mathbf{F}_{ij}^{(i)}(\mathbf{x}_{ij}, t)$ and $\mathbf{F}_{ij}^{(j)}(\mathbf{x}_{ij}, t)$. For a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ for t_m, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$) for an arbitrarily small $\varepsilon > 0$, and $\mathbf{x}_i^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}_j^{(j)}(t_{m+})$. The real flows $\mathbf{x}_i^{(i)}(t)$ and $\mathbf{x}_j^{(j)}(t)$ are $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$- and $C^r_{[t_{m-\varepsilon}, t_m)}$-continuous ($r \geqslant 1$) for time t, respectively. The source bifurcation of the flow $\mathbf{x}_i^{(i)}(t) \cup \mathbf{x}_j^{(j)}(t)$ on the boundary $\overrightarrow{\partial\Omega}_{ij}$ exists iff*

$$\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}_{\text{in}}^{(i)}(t_{m\pm}) = 0 \quad \textit{and} \quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}_{\text{in}}^{(j)}(t_{m+}) \neq 0; \tag{6.27}$$

$$\left.\begin{aligned} &\textit{either} \quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}_{\text{in}}^{(j)}(t_{m+}) > 0 \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ &\textit{or} \quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}_{\text{in}}^{(j)}(t_{m+}) < 0 \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i; \end{aligned}\right. \tag{6.28}$$

$$\begin{aligned} &\textit{either} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathrm{D}\mathbf{F}^{(i)}_{\mathrm{in}}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}_{ij}(t_{m\pm})\big] < 0 \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ &\textit{or} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathrm{D}\mathbf{F}^{(i)}_{\mathrm{in}}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}_{ij}(t_{m\pm})\big] > 0 \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{aligned} \tag{6.29}$$

PROOF. Following the procedure of Theorem 6.1, the imaginary vector fields can be developed. The theorem is proved. □

THEOREM 6.3. *For a discontinuous dynamical system with subsystems in Eq.* (6.1)*, a nonpassable flow on the boundary* $\partial\Omega_{ij}$ *is governed by two flow barriers* $\mathbf{F}^{(i)}_{ij}(\mathbf{x}_{ij}, t)$ *and* $\mathbf{F}^{(j)}_{ij}(\mathbf{x}_{ij}, t)$*. For a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ *for* t_m*, there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$*) for an arbitrarily small* $\varepsilon > 0$*, and* $\mathbf{x}^{(i)}_i(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}_j(t_{m\mp})$*. The real flows* $\mathbf{x}^{(i)}_i(t)$ *and* $\mathbf{x}^{(j)}_j(t)$ *are* $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$*-continuous* ($r \geqslant 1$) *for time* t*. The switching bifurcation of the flow* $\mathbf{x}^{(i)}_i(t) \cup \mathbf{x}^{(j)}_j(t)$ *at* $\mathbf{x}_m$ *on the boundary* $\overrightarrow{\partial\Omega}_{ij}$ *exists iff*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}_{\mathrm{in}}(t_{m\pm}) = 0 \quad \textit{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}_{\mathrm{in}}(t_{m\mp}) = 0, \tag{6.30}$$

$$\textit{either} \quad \left\{ \begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathrm{D}\mathbf{F}^{(i)}_{\mathrm{in}}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}_{ij}(t_{m\pm})\big] < 0 \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathrm{D}\mathbf{F}^{(j)}_{\mathrm{in}}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}_{ij}(t_{m\pm})\big] > 0 \end{aligned} \right. \tag{6.31a}$$
$$\textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$$

$$\textit{or} \quad \left\{ \begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathrm{D}\mathbf{F}^{(i)}_{\mathrm{in}}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}_{ij}(t_{m\pm})\big] > 0 \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathrm{D}\mathbf{F}^{(j)}_{\mathrm{in}}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}_{ij}(t_{m\pm})\big] < 0 \end{aligned} \right. \tag{6.31b}$$
$$\textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i.$$

PROOF. The proof is the same as in Theorem 6.1. □

As in Definition 3.4, the product of the normal vector fields on the boundary $\partial\Omega_{\alpha\beta}$ can be defined for the input flow and the flow barrier, which is given as follows:

DEFINITION 6.4. For a discontinuous dynamical system with subsystems in Eq. (6.1), there is a point $\mathbf{x}_m \in \partial\Omega_{\alpha\beta}$ at t_m for $\alpha, \beta \in \{i, j\}$ and $\alpha \neq \beta$. The *input* and barrier (or output), normal vector fields product on the boundary $\partial\Omega_{ij}$ are defined as

$$L^{(\mathrm{in})}_{\alpha\beta}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_\alpha, \boldsymbol{\mu}_\beta) = \big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}_{\mathrm{in}}(t_{m\mp})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}_{\mathrm{in}}(t_{m\pm})\big], \tag{6.32}$$

$$L^{(\mathrm{b})}_{\alpha\beta}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_\alpha, \boldsymbol{\mu}_\beta) = \big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}_{\mathrm{b}}(t_{m\mp})\big]\big[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}_{\mathrm{b}}(t_{m\pm})\big]. \tag{6.33}$$

From the foregoing definition, the necessary and sufficient conditions for switching bifurcations can be given as in Theorems 3.4–3.6.

THEOREM 6.4. *For a discontinuous dynamical system with subsystems in Eq.* (6.1)*, a nonpassable flow on the boundary* $\partial\Omega_{ij}$ *is governed by two flow barriers* $\mathbf{F}_{ij}^{(i)}(\mathbf{x}_{ij}, t)$ *and* $\mathbf{F}_{ij}^{(j)}(\mathbf{x}_{ij}, t)$. *For a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ *for* t_m*, there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$*) for an arbitrarily small* $\varepsilon > 0$*, and* $\mathbf{x}_i^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}_j^{(j)}(t_{m\pm})$. *The real flows* $\mathbf{x}_i^{(i)}(t)$ *and* $\mathbf{x}_j^{(j)}(t)$ *are* $C^r_{[t_{m-\varepsilon}, t_m)}$*- and* $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 1)$ *for time* t*, respectively. The sliding bifurcation of the flow* $\mathbf{x}_i^{(i)}(t) \cup \mathbf{x}_j^{(j)}(t)$ *on the boundary* $\overrightarrow{\partial\Omega}_{ij}$ *exists iff*

$$L_{ij}^{(\text{in})}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j) = 0 \quad \textit{and} \quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}_{\text{in}}^{(i)}(t_{m-}) \neq 0, \tag{6.34}$$

$$\begin{aligned} &\textit{either} \quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \left[\mathrm{D}\mathbf{F}_{\text{in}}^{(j)}(t_{m\pm}) - \mathrm{D}\mathbf{F}_{ij}^{(0)}(t_{m\pm})\right] > 0 \quad \textit{for}\ \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ &\textit{or} \quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \left[\mathrm{D}\mathbf{F}_{\text{in}}^{(j)}(t_{m\pm}) - \mathrm{D}\mathbf{F}_{ij}^{(0)}(t_{m\pm})\right] < 0 \quad \textit{for}\ \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{aligned} \tag{6.35}$$

PROOF. Application of Eq. (6.28) to Theorem 6.1 gives the foregoing theorem. □

Similarly, we have the following theorems for source bifurcation and switching bifurcations of the discontinuous dynamical system with flow barriers on the separation boundary.

THEOREM 6.5. *For a discontinuous dynamical system with subsystems in Eq.* (6.1)*, a nonpassable flow on the boundary* $\partial\Omega_{ij}$ *is governed by two flow barriers* $\mathbf{F}_{ij}^{(i)}(\mathbf{x}_{ij}, t)$ *and* $\mathbf{F}_{ij}^{(j)}(\mathbf{x}_{ij}, t)$. *For a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ *for* t_m*, there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$*) for an arbitrarily small* $\varepsilon > 0$*, and* $\mathbf{x}_i^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}_j^{(j)}(t_{m+})$. *The real flows* $\mathbf{x}_i^{(i)}(t)$ *and* $\mathbf{x}_j^{(j)}(t)$ *are* $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$*- and* $C^r_{[t_{m-\varepsilon}, t_m)}$*-continuous* $(r \geqslant 1)$ *for time* t*, respectively. The source bifurcation of the flow* $\mathbf{x}_i^{(i)}(t) \cup \mathbf{x}_j^{(j)}(t)$ *on the boundary* $\overrightarrow{\partial\Omega}_{ij}$ *exists iff*

$$L_{ij}^{(\text{in})}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j) = 0 \quad \textit{and} \quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}^{(j)}(t_{m+}) \neq 0, \tag{6.36}$$

$$\begin{aligned} &\textit{either} \quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \left[\mathrm{D}\mathbf{F}_{\text{in}}^{(i)}(t_{m\pm}) - \mathrm{D}\mathbf{F}_{ij}^{(0)}(t_{m\pm})\right] < 0 \quad \textit{for}\ \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ &\textit{or} \quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \left[\mathrm{D}\mathbf{F}_{\text{in}}^{(i)}(t_{m\pm}) - \mathrm{D}\mathbf{F}_{ij}^{(0)}(t_{m\pm})\right] > 0 \quad \textit{for}\ \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{aligned} \tag{6.37}$$

THEOREM 6.6. *For a discontinuous dynamical system with subsystems in Eq.* (6.1)*, a nonpassable flow on the boundary* $\partial\Omega_{ij}$ *is governed by two flow barriers* $\mathbf{F}_{ij}^{(i)}(\mathbf{x}_{ij}, t)$ *and* $\mathbf{F}_{ij}^{(j)}(\mathbf{x}_{ij}, t)$. *For a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ *for*

t_m, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$) for an arbitrarily small $\varepsilon > 0$, and $\mathbf{x}_i^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}_j^{(j)}(t_{m\mp})$. The real flows $\mathbf{x}_i^{(i)}(t)$ and $\mathbf{x}_j^{(j)}(t)$ are $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous ($r \geqslant 1$) for time t. The switching bifurcation of the flow $\mathbf{x}_i^{(i)}(t) \cup \mathbf{x}_j^{(j)}(t)$ at $\mathbf{x}_m$ on the boundary $\overrightarrow{\partial\Omega}_{ij}$ exists iff for $\alpha \in \{i, j\}$

$$L_{ij}^{(\text{in})}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j) = 0 \quad \textit{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m\pm}) = 0, \tag{6.38}$$

$$\begin{aligned} &\textit{either} \quad \begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathrm{D}\mathbf{F}^{(i)}_{\text{in}}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}_{ij}(t_{m\pm})\right] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathrm{D}\mathbf{F}^{(j)}_{\text{in}}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}_{ij}(t_{m\pm})\right] > 0 \end{cases} \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ &\textit{or} \quad \begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathrm{D}\mathbf{F}^{(i)}_{\text{in}}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}_{ij}(t_{m\pm})\right] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathrm{D}\mathbf{F}^{(j)}_{\text{in}}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}_{ij}(t_{m\pm})\right] < 0 \end{cases} \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{aligned} \tag{6.39}$$

PROOF. Application of Eq. (6.29) to Theorem 6.3 proves theorem. □

Similarly to ${}_{\text{Gmin}}L_{ij}(t_m)$ in Definition 3.6, ${}_{\text{Gmin}}L_{ij}^{(\text{in})}(t_m)$ can be defined. The corollaries of previous theorems will be obtained when ${}_{\text{Gmin}}L_{ij}^{(\text{in})}(t_m)$ replaces ${}_{\text{Gmin}}L_{ij}(t_m)$ in Corollaries 3.1–3.3. The sufficient and necessary conditions for the onset of the sliding, source and switching bifurcations on the boundary $\overrightarrow{\partial\Omega}_{ij}$ are obtained accordingly. In other words, when the condition ${}_{\text{Gmin}}L_{ij}^{(\text{in})}(t_m) = 0$ replaces the conditions in Eqs. (6.34), (6.36) and (6.38) of Theorems 6.4–6.6, such bifurcation onset on the boundary $\overrightarrow{\partial\Omega}_{ij}$ will be obtained.

6.3. Sliding fragmentation

As discussed in Section 6.1, once a nonpassable flow on the boundary is formed, the flow will be controlled by the flow barrier on the separation boundary. Therefore, the vector fields will be replaced by the lower and upper flow barriers $\mathbf{F}_{\text{b}}^{(\alpha)}(t_{m\pm})$ ($\alpha \in \{i, j\}$). Further, the necessary and sufficient conditions for the sliding and source fragmentations are presented as follows.

THEOREM 6.7. *For a discontinuous dynamical system with subsystems in Eq. (6.1), a nonpassable flow on the boundary $\partial\Omega_{ij}$ is governed by two flow barriers $\mathbf{F}_{ij}^{(i)}(\mathbf{x}_{ij}, t)$ and $\mathbf{F}_{ij}^{(j)}(\mathbf{x}_{ij}, t)$. For a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widetilde{\partial\Omega}_{ij}$ for t_m, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$) for an arbitrarily small $\varepsilon > 0$, and $\mathbf{x}_\alpha^{(\alpha)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}_\beta^{(\beta)}(t_{m\pm})$, $\alpha, \beta \in \{i, j\}$ and $\alpha \neq \beta$. The real flows $\mathbf{x}_\alpha^{(\alpha)}(t)$ and $\mathbf{x}_\beta^{(\beta)}(t)$ are $C^r_{[t_{m-\varepsilon}, t_m)}$- and $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous ($r \geqslant 1$) for time t, respectively. The sliding fragmentation bifurcation of the* *flow* $\mathbf{x}_\alpha^{(\alpha)}(t) \cup \mathbf{x}_\beta^{(\beta)}(t)$ *on the boundary* $\partial\widetilde{\Omega}_{ij}$ *exists iff*

$$\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}_{\mathrm{b}}^{(\beta)}(t_{m\pm}) = 0 \quad \textit{and} \quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}_{\mathrm{b}}^{(\alpha)}(t_{m-}) \neq 0; \tag{6.40}$$

$$\begin{aligned} &\textit{either} \quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}_{\mathrm{b}}^{(\alpha)}(t_{m-}) > 0 \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j, \\ &\textit{or} \quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}_{\mathrm{b}}^{(\alpha)}(t_{m-}) < 0 \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i; \end{aligned} \tag{6.41}$$

$$\begin{aligned} &\textit{either} \quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \left[\mathrm{D}\mathbf{F}_{\mathrm{b}}^{(\beta)}(t_{m\pm}) - \mathrm{D}\mathbf{F}_{\alpha\beta}^{(0)}(t_{m\pm})\right] > 0 \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ &\textit{or} \quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \left[\mathrm{D}\mathbf{F}_{\mathrm{b}}^{(\beta)}(t_{m\pm}) - \mathrm{D}\mathbf{F}_{\alpha\beta}^{(0)}(t_{m\pm})\right] < 0 \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{aligned} \tag{6.42}$$

PROOF. Based on the flow barriers $\mathbf{F}_{\alpha\beta}^{(\alpha)}(\mathbf{x}_{\alpha\beta}, t)$ and $\mathbf{F}_{\alpha\beta}^{(\beta)}(\mathbf{x}_{\alpha\beta}, t)$ ($\alpha, \beta \in \{i, j\}$, $\alpha \neq \beta$) on the boundary $\partial\Omega_{ij}$, construct two imaginary vector fields in the domain Ω_α and Ω_β as

$$\begin{aligned} \mathbf{F}_{\mathrm{b}}^{(\alpha)}\left(\mathbf{x}_\alpha^{(\alpha)}, t, \boldsymbol{\mu}_\alpha\right) &\equiv \mathbf{F}_{\alpha\beta}^{(\alpha)}\left(\mathbf{x}_\alpha^{(\alpha)}, t, \boldsymbol{\mu}_\alpha\right) + \mathbf{g}(\mathbf{x}, t), \\ \mathbf{F}_{\mathrm{b}}^{(\beta)}\left(\mathbf{x}_\beta^{(\beta)}, t, \boldsymbol{\mu}_\beta\right) &\equiv \mathbf{F}_{\alpha\beta}^{(\beta)}\left(\mathbf{x}_\beta^{(\beta)}, t, \boldsymbol{\mu}_\beta\right) + \mathbf{g}(\mathbf{x}, t). \end{aligned}$$

For a point $\mathbf{x}_m \in \partial\Omega_{\alpha\beta}$ at time t_m, as time $t \to t_{m-}$, the flow $\mathbf{x}_\alpha^{(\alpha)} \to \mathbf{x}_m$ and the corresponding vector field $\mathbf{F}_{\mathrm{b}}^{(\alpha)}(\mathbf{x}_\alpha^{(\alpha)}, t, \boldsymbol{\mu}_\alpha) \to \mathbf{F}_{\mathrm{b}}^{(\alpha)}(\mathbf{x}_m, t_{m-}, \boldsymbol{\mu}_\alpha)$. From Definition 6.2, $\mathbf{F}_{\mathrm{b}}^{(\alpha)}(\mathbf{x}_m, t_{m-}, \boldsymbol{\mu}_\alpha) = \mathbf{F}_{\mathrm{b}}^{(\alpha)}(t_{m-})$. Such relations are suitable for the imaginary vector field in the domain Ω_β. Further, the imaginary discontinuous dynamic system is given by

$$\dot{\mathbf{x}}_\alpha^{(\alpha)} = \mathbf{F}_{\mathrm{b}}^{(\alpha)}\left(\mathbf{x}_\alpha^{(\alpha)}, t, \boldsymbol{\mu}_\alpha\right), \quad \alpha \in \{i, j\}.$$

Using Definition 3.7 for the sliding fragmentation, following the proof procedure of Theorems 2.10–2.11, Eqs. (6.41)–(6.42) can be obtained. □

THEOREM 6.8. *For a discontinuous dynamical system with subsystems in Eq. (6.1), a nonpassable flow on the boundary* $\partial\Omega_{ij}$ *is governed by two flow barriers* $\mathbf{F}_{ij}^{(i)}(\mathbf{x}_{ij}, t)$ *and* $\mathbf{F}_{ij}^{(j)}(\mathbf{x}_{ij}, t)$. *For a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \partial\widehat{\Omega}_{ij}$ *for* t_m, *there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$*) for an arbitrarily small* $\varepsilon > 0$, *and* $\mathbf{x}_\alpha^{(\alpha)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}_\beta^{(\beta)}(t_{m\pm})$, $\alpha, \beta \in \{i, j\}$ *and* $\alpha \neq \beta$. *The real flows* $\mathbf{x}_\alpha^{(\alpha)}(t)$ *and* $\mathbf{x}_\beta^{(\beta)}(t)$ *are* $C^r_{[t_{m-\varepsilon}, t_m)}$*- and* $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$*-continuous* ($r \geqslant 1$) *for time* t, *respectively. The source fragmentation bifurcation of the flow* $\mathbf{x}_\alpha^{(\alpha)}(t) \cup \mathbf{x}_\beta^{(\beta)}(t)$ *on the boundary* $\partial\widehat{\Omega}_{ij}$ *occurs iff*

$$\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}_{\mathrm{b}}^{(\alpha)}(t_{m\pm}) = 0 \quad \textit{and} \quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}_{\mathrm{b}}^{(\beta)}(t_{m+}) \neq 0; \tag{6.43}$$

$$\begin{aligned} &\textit{either} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}_{\mathrm{b}}(t_{m+}) > 0 \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j, \\ &\textit{or} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}_{\mathrm{b}}(t_{m+}) < 0 \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i; \end{aligned} \tag{6.44}$$

$$\begin{aligned} &\textit{either} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathrm{D}\mathbf{F}^{(\alpha)}_{\mathrm{b}}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}_{\alpha\beta}(t_{m\pm})\big] < 0 \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ &\textit{or} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathrm{D}\mathbf{F}^{(\alpha)}_{\mathrm{b}}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}_{\alpha\beta}(t_{m\pm})\big] > 0 \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{aligned} \tag{6.45}$$

PROOF. As in Theorem 6.7, the imaginary vector fields and imaginary discontinuous dynamic systems can be constructed through the flow barrier vector fields. Further, this theorem is proved as Theorem 6.7. □

THEOREM 6.9. *For a discontinuous dynamical system with subsystems in Eq. (6.1), a nonpassable flow on the boundary $\partial\Omega_{ij}$ is governed by two flow barriers $\mathbf{F}^{(i)}_{ij}(\mathbf{x}_{ij}, t)$ and $\mathbf{F}^{(j)}_{ij}(\mathbf{x}_{ij}, t)$. For a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widetilde{\partial\Omega}_{ij}$ (or $\widehat{\partial\Omega}_{ij}$) for t_m, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$) for an arbitrarily small $\varepsilon > 0$, and $\mathbf{x}^{(\alpha)}_{\alpha}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(\beta)}_{\beta}(t_{m\pm})$, $\alpha, \beta \in \{i, j\}$ and $\alpha \neq \beta$. The real flows $\mathbf{x}^{(\alpha)}_{\alpha}(t)$ and $\mathbf{x}^{(\beta)}_{\beta}(t)$ are $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous ($r \geqslant 1$) for time t. The switching bifurcation of the real flow from $\widetilde{\partial\Omega}_{ij}$ to $\widehat{\partial\Omega}_{ij}$ (or $\widehat{\partial\Omega}_{ij}$ to $\widetilde{\partial\Omega}_{ij}$) exists iff*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}_{\mathrm{b}}(t_{m\pm}) = 0 \quad \textit{for } \alpha \in \{i, j\}, \tag{6.46}$$

$$\begin{aligned} &\textit{either} \quad \left\{ \begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathrm{D}\mathbf{F}^{(i)}_{\mathrm{b}}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}_{\alpha\beta}(t_{m\pm})\big] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathrm{D}\mathbf{F}^{(j)}_{\mathrm{b}}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}_{\alpha\beta}(t_{m\pm})\big] > 0 \end{array} \right. \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ &\textit{or} \quad \left\{ \begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathrm{D}\mathbf{F}^{(i)}_{\mathrm{b}}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}_{\alpha\beta}(t_{m\pm})\big] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \big[\mathrm{D}\mathbf{F}^{(j)}_{\mathrm{b}}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}_{\alpha\beta}(t_{m\pm})\big] < 0 \end{array} \right. \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{aligned} \tag{6.47}$$

PROOF. As in Theorem 6.7, the imaginary vector field and imaginary discontinuous dynamic systems can be constructed through the flow barrier vector fields. Further, this theorem is proved as Theorem 6.9. □

By using Eq. (6.33), the necessary and sufficient conditions for onset of sliding fragmentation on the boundary in discontinuous dynamical systems with flow barriers can be given as in Theorems 3.4–3.6.

THEOREM 6.10. *For a discontinuous dynamical system with subsystems in Eq. (6.1), a nonpassable flow on the boundary $\partial\Omega_{ij}$ is governed by two flow barriers $\mathbf{F}^{(i)}_{ij}(\mathbf{x}_{ij}, t)$ and $\mathbf{F}^{(j)}_{ij}(\mathbf{x}_{ij}, t)$. For a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widetilde{\partial\Omega}_{ij}$ for*

t_m, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$) for an arbitrarily small $\varepsilon > 0$, and $\mathbf{x}_\alpha^{(\alpha)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}_\beta^{(\beta)}(t_{m\pm})$, $\alpha, \beta \in \{i, j\}$ and $\alpha \neq \beta$. The real flows $\mathbf{x}_\alpha^{(\alpha)}(t)$ and $\mathbf{x}_\beta^{(\beta)}(t)$ are $C^r_{[t_{m-\varepsilon}, t_m)}$- and $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous $(r \geqslant 1)$ for time t, respectively. The sliding fragmentation bifurcation of the flow $\mathbf{x}_\alpha^{(\alpha)}(t) \cup \mathbf{x}_\beta^{(\beta)}(t)$ on the boundary $\widetilde{\partial\Omega}_{ij}$ exists iff

$$L_{\alpha\beta}^{(\mathrm{b})}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_\alpha, \boldsymbol{\mu}_\beta) = 0 \quad \textit{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}_{\mathrm{b}}^{(\alpha)}(t_{m+}) \neq 0, \tag{6.48}$$

$$\begin{aligned} &\textit{either} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathrm{D}\mathbf{F}_{\mathrm{b}}^{(\beta)}(t_{m\pm}) - \mathrm{D}\mathbf{F}_{\alpha\beta}^{(0)}(t_{m\pm})\right] > 0 \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta \\ &\textit{or} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathrm{D}\mathbf{F}_{\mathrm{b}}^{(\beta)}(t_{m\pm}) - \mathrm{D}\mathbf{F}_{\alpha\beta}^{(0)}(t_{m\pm})\right] < 0 \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha. \end{aligned} \tag{6.49}$$

PROOF. Application of Eq. (6.33) to Theorem 6.7 gives the foregoing theorem. □

THEOREM 6.11. *For a discontinuous dynamical system with subsystems in Eq. (6.1), a nonpassable flow on the boundary $\partial\Omega_{ij}$ is governed by two flow barriers $\mathbf{F}_{ij}^{(i)}(\mathbf{x}_{ij}, t)$ and $\mathbf{F}_{ij}^{(j)}(\mathbf{x}_{ij}, t)$. For a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widehat{\partial\Omega}_{ij}$ for t_m, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$) for an arbitrarily small $\varepsilon > 0$, and $\mathbf{x}_\alpha^{(\alpha)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}_\beta^{(\beta)}(t_{m\pm})$, $\alpha, \beta \in \{i, j\}$ and $\alpha \neq \beta$. The real flows $\mathbf{x}_\alpha^{(\alpha)}(t)$ and $\mathbf{x}_\beta^{(\beta)}(t)$ are $C^r_{[t_{m-\varepsilon}, t_m)}$- and $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous $(r \geqslant 1)$ for time t, respectively. The source fragmentation bifurcation of the flow $\mathbf{x}_\alpha^{(\alpha)}(t) \cup \mathbf{x}_\beta^{(\beta)}(t)$ on the boundary $\widehat{\partial\Omega}_{ij}$ occurs iff*

$$L_{\alpha\beta}^{(\mathrm{b})}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_\alpha, \boldsymbol{\mu}_\beta) = 0 \quad \textit{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}(t_{m+}) \neq 0, \tag{6.50}$$

$$\begin{aligned} &\textit{either} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathrm{D}\mathbf{F}_{\mathrm{b}}^{(\alpha)}(t_{m\pm}) - \mathrm{D}\mathbf{F}_{\alpha\beta}^{(0)}(t_{m\pm})\right] < 0 \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta \\ &\textit{or} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathrm{D}\mathbf{F}_{\mathrm{b}}^{(\alpha)}(t_{m\pm}) - \mathrm{D}\mathbf{F}_{\alpha\beta}^{(0)}(t_{m\pm})\right] > 0 \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha. \end{aligned} \tag{6.51}$$

PROOF. Application of Eq. (6.33) to Theorem 6.8 gives the foregoing theorem. □

THEOREM 6.12. *For a discontinuous dynamical system with subsystems in Eq. (6.1), a nonpassable flow on the boundary $\partial\Omega_{ij}$ is governed by two flow barriers $\mathbf{F}_{ij}^{(i)}(\mathbf{x}_{ij}, t)$ and $\mathbf{F}_{ij}^{(j)}(\mathbf{x}_{ij}, t)$. For a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widetilde{\partial\Omega}_{ij}$ (or $\widehat{\partial\Omega}_{ij}$) for t_m, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$) for an arbitrarily small $\varepsilon > 0$, and $\mathbf{x}_\alpha^{(\alpha)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}_\beta^{(\beta)}(t_{m\pm})$, $\alpha, \beta \in \{i, j\}$ and $\alpha \neq \beta$. The real flows $\mathbf{x}_\alpha^{(\alpha)}(t)$ and $\mathbf{x}_\beta^{(\beta)}(t)$ are $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous $(r \geqslant 1)$ for time t. The switching bifurcation of the real flow from $\widetilde{\partial\Omega}_{ij}$ to $\widehat{\partial\Omega}_{ij}$ (or $\widehat{\partial\Omega}_{ij}$ to*

$\widetilde{\partial\Omega}_{ij}$) *exists iff*

$$L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j) = 0, \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m\pm}) = 0 \quad \textit{and} \tag{6.52}$$

$$\begin{aligned} &\textit{either} \quad \begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathrm{D}\mathbf{F}^{(i)}_{\mathrm{b}}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}_{ij}(t_{m\pm})\right] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathrm{D}\mathbf{F}^{(j)}_{\mathrm{b}}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}(t_{m\pm})\right] > 0 \end{cases} \quad \textit{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ &\textit{or} \quad \begin{cases} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathrm{D}\mathbf{F}^{(i)}_{\mathrm{b}}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}(t_{m\pm})\right] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathrm{D}\mathbf{F}^{(j)}_{\mathrm{b}}(t_{m\pm}) - \mathrm{D}\mathbf{F}^{(0)}(t_{m\pm})\right] < 0 \end{cases} \quad \textit{or } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{aligned} \tag{6.53}$$

PROOF. Application of Eq. (6.29) to Theorem 6.9 proves the theorem. □

As in Definitions 3.10 and 3.11, ${}_{\mathrm{Gmax}}L^{(\mathrm{b})}_{ij}(t_m)$ can be defined. The corollaries of the Theorems 6.10–6.12 will be obtained when ${}_{\mathrm{Gmax}}L^{(\mathrm{b})}_{ij}(t_m)$ replaces ${}_{\mathrm{Gmax}}L_{ij}(t_m)$ in Corollaries 3.4–3.6. The sufficient and necessary conditions for the onset of the sliding, source and fragmentation bifurcations on the boundary $\widetilde{\partial\Omega}_{ij}$ (or $\widehat{\partial\Omega}_{ij}$) are obtained accordingly. When the condition ${}_{\mathrm{Gmax}}L^{(\mathrm{b})}_{ij}(t_m) = 0$ replaces the condition in Eqs. (6.48), (6.50) and (6.52) of the three theorems, such bifurcation onset on the boundary $\widetilde{\partial\Omega}_{ij}$ (or $\widehat{\partial\Omega}_{ij}$) will be obtained.

6.4. A friction oscillator with flow barrier

Consider a dynamic system with a nonlinear friction, but the nonlinear friction is approximated by a piecewise linear model, as shown in Fig. 6.3. The dynamic system consists of a mass, m, a spring of stiffness, k, and a damper of viscous damping coefficient, r. The oscillator mass rests on the horizontal belt surface traveling with a constant speed, V. The coordinate system (x, t) is absolute with displacement x and time t. The periodic excitation force $Q_0 \cos \Omega t$ exerts on the mass, where Q_0 and Ω are the excitation strength and frequency, respectively. The piecewise linear friction force is given by

$$\overline{F}_f(\dot{x}) \begin{cases} = \mu_1(\dot{x} - V_1) - \mu_2(V_1 - V) + F_N\mu_k, & \dot{x} \in [V_1, \infty), \\ = -\mu_2(\dot{x} - V) + F_N\mu_k, & \dot{x} \in (V, V_1), \\ \in [-\mu_s F_N, \mu_s F_N], & \dot{x} = V, \\ = -\mu_3(\dot{x} - V) - F_N\mu_k, & \dot{x} \in (V_2, V), \\ = \mu_4(\dot{x} - V_2) - \mu_3(V_2 - V) - F_N\mu_k, & \dot{x} \in (-\infty, V_2] \end{cases} \tag{6.54}$$

where $\dot{x} \triangleq \mathrm{d}x/\mathrm{d}t$, the parameters ($\mu_s$, μ_k and F_N) are static and kinetic friction coefficients and a normal force to the contact surface, respectively. The coefficients μ_j $(j = 1, 2, 3, 4)$ are the slopes for friction force with velocity. For this problem, $F_N = mg$ and g is the gravitational acceleration. In fact, this model can

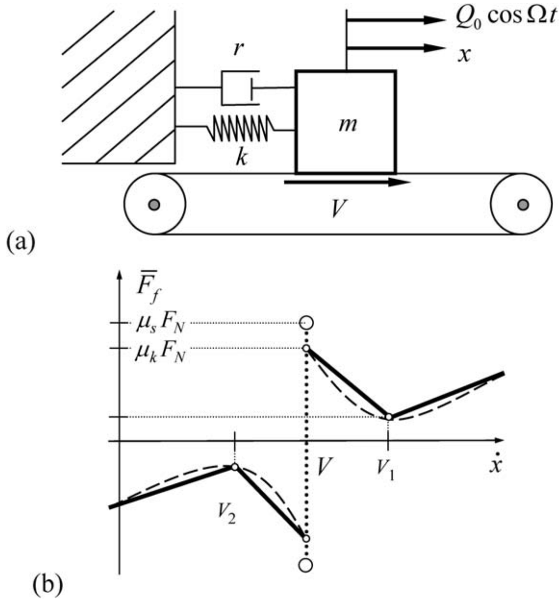


Figure 6.3. (a) Schematic mechanical model of the friction-induced oscillator, and (b) piecewise linear friction force model.

be applied to many mechanical problems, and the normal force can be exerted externally. The nonfriction force per unit mass acting on the mass in the x-direction is determined by

$$F_s = A_0 \cos \Omega t - 2dV - cx, \quad \text{for } \dot{x} = V \tag{6.55}$$

where $A_0 = Q_0/m$, $d = r/2m$ and $c = k/m$. For a stick motion, the nonfriction force is less than the static friction force, i.e., $|F_s| \leqslant F_{f_s}$ and $F_{f_s} = \mu_s F_N/m$. The mass does not have any relative motion to the belt. Therefore, no acceleration exists because the belt speed is constant, i.e.,

$$\ddot{x} = 0, \quad \text{for } \dot{x} = V. \tag{6.56}$$

If the nonfriction force is greater than the static friction force on the mass (i.e., $|F_s| > F_{f_s}$), the nonstick motion occurs. For a nonstick motion, the total force acting on the mass is

$$F = A_0 \cos \Omega t - F_{f_k} \text{sgn}(\dot{x} - V) - 2d\dot{x} - cx, \quad \text{for } \dot{x} \neq V; \tag{6.57}$$

where $F_{f_k} = \mu_k F_N/m$. Therefore, the equation of the nonstick motion for this dynamical system with a piecewise linear friction is

$$\ddot{x} + 2d\dot{x} + cx = A_0 \cos \Omega t - F_{f_k} \text{sgn}(\dot{x} - V), \quad \text{for } \dot{x} \neq V. \tag{6.58}$$

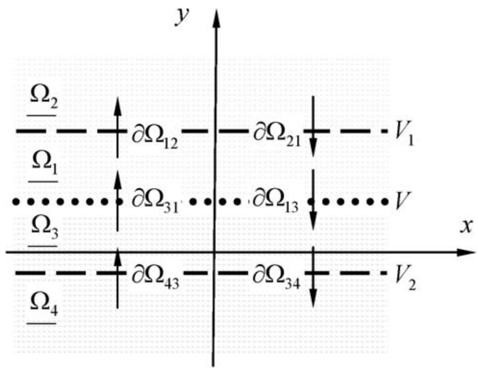


Figure 6.4. Phase plane partition and oriented boundaries.

The discontinuities of this dynamical system are caused by a jumping from static to dynamic friction forces and piecewise linear dynamical friction model. So the phase plane should be partitioned into four domains. The friction force jumping will be as a main discontinuity.

Therefore the naming of the phase domains starts from the domain near the main discontinuous boundary $\dot{x} = V$. Based on the direction of trajectories of mass motion, the corresponding boundaries are named, as shown in Fig. 6.4. The friction force jumping boundary and the rest boundaries are depicted by the dotted and dashed lines, respectively. The domain naming can be arbitrarily. The four regions are expressed by Ω_j $(j = 1, 2, 3, 4)$.

In phase plane, the flow and vector field for such a system are introduced as

$$\mathbf{x} \triangleq (x, \dot{x})^{\mathrm{T}} \equiv (x, y)^{\mathrm{T}} \quad \text{and} \quad \mathbf{F} \triangleq (y, F)^{\mathrm{T}}. \tag{6.59}$$

The named domains and the oriented boundaries are expressed by

$$\begin{aligned} \Omega_1 &= \{(x, y) \mid y \in (V, V_1)\}, \quad \Omega_2 = \{(x, y) \mid y \in (V_1, \infty)\}, \\ \Omega_3 &= \{(x, y) \mid y \in (V_2, V)\}, \quad \Omega_4 = \{(x, y) \mid y \in (-\infty, V_2)\}; \end{aligned} \tag{6.60}$$

$$\partial\Omega_{ij} = \{(x, y) \mid \varphi_{ij}(x, y) \equiv y - V_\rho = 0\}, \tag{6.61}$$

where $\rho = 1$ if $i, j \in \{1, 2\}$, $\rho = 0$ if $i, j \in \{1, 3\}$, and $\rho = 2$ if $i, j \in \{3, 4\}$. $V_0 \triangleq V$. The subscripts $(\cdot)_{ij}$ define the boundary from Ω_i to Ω_j. The domains are accessible for a certain vector field. On the boundary $\partial\Omega_{13}$ or $\partial\Omega_{31}$, the vector fields are C^0-discontinuous, but on the boundaries $\partial\Omega_{12}$ and $\partial\Omega_{34}$, the vector fields are C^0-continuous.

The equation of motion in Eqs. (6.56) and (6.58) is described as

$$\dot{\mathbf{x}} = \mathbf{F}^{(j)}(\mathbf{x}, t), \quad j \in \{0, 1, 2, 3, 4\} \tag{6.62}$$

where

$$\begin{aligned}\mathbf{F}^{(0)}(\mathbf{x},t) &= (V,0)^{\mathrm{T}} \quad \text{on } \partial\Omega_{13} \text{ or } \partial\Omega_{31},\\ \mathbf{F}^{(j)}(\mathbf{x},t) &= \big(y, F_j(\mathbf{x},t)\big)^{\mathrm{T}} \quad \text{in } \Omega_j \ (j\in\{1,2,3,4\}),\end{aligned} \tag{6.63}$$

$$F_j(\mathbf{x},t) = A_0\cos\Omega t - F_{f_k}^{(j)}(\mathbf{x},t) - 2d_j y - c_j x. \tag{6.64}$$

From Eq. (6.54), the dynamical friction forces are defined as

$$\begin{aligned}F_{f_k}^{(2)}(\mathbf{x},t) &= \mu_1(\dot{x}-V_1) - \mu_2(V_1-V) + F_N\mu_k, && \dot{x}\in[V_1,\infty),\\ F_{f_k}^{(1)}(\mathbf{x},t) &= -\mu_2(\dot{x}-V) + F_N\mu_k, && \dot{x}\in(V,V_1),\\ F_{f_k}^{(3)}(\mathbf{x},t) &= -\mu_3(\dot{x}-V) - F_N\mu_k, && \dot{x}\in(V_2,V),\\ F_{f_k}^{(4)}(\mathbf{x},t) &= \mu_4(\dot{x}-V_2) - \mu_3(V_2-V) - F_N\mu_k, && \dot{x}\in(-\infty,V_2].\end{aligned} \tag{6.65}$$

The two force boundaries relative to $V_{1,2}$ (i.e., $\dot{x} = V_1$ or V_2) are C^0-continuous. However, the boundary relative to the velocity V is a discontinuous force boundary. Because the input flow barrier does not exist, the input flow vector fields on the boundary $\partial\Omega_{13}$ are defined for $\mathbf{x}_m \in \partial\Omega_{\alpha\beta}$ ($\alpha,\beta\in\{1,3\}$)

$$\mathbf{F}_{\text{in}}^{(\alpha)}(\mathbf{x}_m,t_m) = \big(y, F_{\text{in}}^{(\alpha)}(\mathbf{x}_m,t)\big)^{\mathrm{T}} \quad \text{and} \quad F_{\text{in}}^{(\alpha)}(\mathbf{x}_m,t_{m-}) \equiv F_\alpha(\mathbf{x}_m,t). \tag{6.66}$$

The time t_m represents the moment for the motion just on the separation boundary and the time $t_{m\pm} = t_m \pm 0$ reflects the flows in the regions instead of the separation boundary. However, the output flow barriers on the boundary $\partial\Omega_{13}$ are for $\mathbf{x}_m \in \partial\Omega_{\alpha\beta}$ ($\alpha,\beta\in\{1,3\}$)

$$\begin{aligned}\mathbf{F}_{\alpha\beta}^{(\alpha)}(\mathbf{x}_m,t_{m-}) &= \big(y_m, F_{\alpha\beta}^{(\alpha)}(\mathbf{x}_m,t_{m-})\big)^{\mathrm{T}},\\ F_{\alpha\beta}^{(\alpha)}(\mathbf{x}_m,t_m) &= F_{f_s}^{(\alpha)} - 2d_\alpha y_m - c_\alpha x_m.\end{aligned} \tag{6.67}$$

Again, from Eq. (6.54), the static frictional forces are defined on the boundary $\partial\Omega_{13}$ as

$$F_{f_s}^{(1)} = +F_N\mu_k \quad \text{and} \quad F_{f_s}^{(3)} = -F_N\mu_k. \tag{6.68}$$

Furthermore, the output flow vector fields with the output flow barriers on the boundary the boundary $\partial\Omega_{13}$ are for $\mathbf{x}_m \in \partial\Omega_{\alpha\beta}$ ($\alpha,\beta\in\{1,3\}$)

$$\begin{aligned}\mathbf{F}_{\mathrm{b}}^{(\alpha)}(\mathbf{x}_m,t_{m-}) &= \big(y, F_{\mathrm{b}}^{(\alpha)}(\mathbf{x}_m,t)\big)^{\mathrm{T}},\\ F_{\mathrm{b}}^{(\alpha)}(\mathbf{x}_m,t_{m-}) &\equiv A_0\cos\Omega t_m + F_{\alpha\beta}^{(\alpha)}(\mathbf{x}_m,t_m)\\ &= A_0\cos\Omega t_m - F_{f_k}^{(\alpha)} - 2d_j y_m - c_j x_m.\end{aligned} \tag{6.69}$$

From the nonsmooth dynamics theory in Luo (2005a), only on the discontinuous force boundary $\partial\Omega_{\alpha\beta}$ ($\alpha, \beta \in \{1, 3\}$), the stick motion for $\partial\Omega_{\alpha\beta}$ convex to Ω_α is guaranteed by

$$\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(\mathbf{x}_m, t_{m-})\right] < 0 \quad \text{and} \quad \left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}(\mathbf{x}_m, t_{m-})\right] > 0 \tag{6.70}$$

where $\alpha, \beta \in \{1, 3\}$ and $\alpha \neq \beta$ with

$$\mathbf{n}_{\partial\Omega_{\alpha\beta}} = \nabla\varphi_{\alpha\beta} = \left(\frac{\partial\varphi_{\alpha\beta}}{\partial x}, \frac{\partial\varphi_{\alpha\beta}}{\partial y}\right)^{\mathrm{T}}_{(x_m, y_m)}. \tag{6.71}$$

Notice that $\nabla = \partial/\partial x\mathbf{i} + \partial/\partial y\mathbf{j}$ is the Hamilton operator. However, the corresponding necessary and sufficient conditions for the nonstick motion (or passable motion) are for $\partial\Omega_{\alpha\beta}$ convex to Ω_α

$$\begin{aligned} &\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(\mathbf{x}_m, t_{m-})\right] < 0 \quad \text{and} \quad \left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}(\mathbf{x}_m, t_{m+})\right] < 0 \\ &\quad \text{for } \Omega_\alpha \to \Omega_\beta, \\ &\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\beta\alpha}} \cdot \mathbf{F}^{(\beta)}(\mathbf{x}_m, t_{m-})\right] > 0 \quad \text{and} \quad \left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\beta\alpha}} \cdot \mathbf{F}^{(\alpha)}(\mathbf{x}_m, t_{m+})\right] > 0 \\ &\quad \text{for } \Omega_\beta \to \Omega_\alpha. \end{aligned} \tag{6.72}$$

The second equation gives the necessary and sufficient conditions for the motion without sliding. Once the sliding motion exists, the flow will overcome the flow barrier into the domain Ω_α ($\alpha = \{1, 3\}$).

Using Eq. (6.61), Eq. (6.69) gives for $i, j \in \{1, 2, 3, 4\}$

$$\mathbf{n}_{\partial\Omega_{ij}} = \mathbf{n}_{\partial\Omega_{ji}} = (0, 1)^{\mathrm{T}}. \tag{6.73}$$

The boundaries ($\partial\Omega_{12}$ and $\partial\Omega_{21}$), ($\partial\Omega_{13}$ and $\partial\Omega_{31}$) and ($\partial\Omega_{34}$ and $\partial\Omega_{43}$) are convex to Ω_2, Ω_1 and Ω_3, respectively. Therefore, we have

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ji}} \cdot \mathbf{F}^{(j)}(t) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t) = F_j(\mathbf{x}, t). \tag{6.74}$$

With Eq. (6.74), the conditions for stick and nonstick motions in Eqs. (6.70) and (6.72) become:

$$\begin{aligned} &F_1(\mathbf{x}_m, t_{m-}) < 0 \quad \text{and} \quad F_3(\mathbf{x}_m, t_{m-}) > 0 \quad \text{on } \partial\Omega_{13}; \\ &F_1(\mathbf{x}_m, t_{m-}) < 0 \quad \text{and} \quad F_3(\mathbf{x}_m, t_{m+}) < 0 \quad \text{for } \Omega_1 \to \Omega_3, \\ &F_1(\mathbf{x}_m, t_{m+}) > 0 \quad \text{and} \quad F_3(\mathbf{x}_m, t_{m-}) > 0 \quad \text{for } \Omega_3 \to \Omega_1. \end{aligned} \tag{6.75}$$

Owing to the flow barrier, from Theorem 6.7, the force conditions for vanishing of the stick motions are

$$\begin{aligned} &F^{(1)}_{\mathrm{b}}(\mathbf{x}_m, t_{m-}) < 0 \quad \text{and} \quad F^{(3)}_{\mathrm{b}}(\mathbf{x}_m, t_{m-}) = 0 \quad \text{for } \partial\Omega_{13} \to \Omega_3; \\ &F^{(3)}_{\mathrm{b}}(\mathbf{x}_m, t_{m-}) > 0 \quad \text{and} \quad F^{(3)}_{\mathrm{b}}(\mathbf{x}_m, t_{m-}) = 0 \quad \text{for } \partial\Omega_{13} \to \Omega_1. \end{aligned} \tag{6.76}$$

The onset condition for the stick motion is

$$\begin{aligned} &F_{\text{in}}^{(1)}(\mathbf{x}_m, t_{m-}) < 0 \quad \text{and} \quad F_{\text{in}}^{(3)}(\mathbf{x}_m, t_{m+}) = 0 \quad \text{for } \Omega_1 \to \partial\Omega_{13}; \\ &F_{\text{in}}^{(3)}(\mathbf{x}_m, t_{m-}) > 0 \quad \text{and} \quad F_{\text{in}}^{(1)}(\mathbf{x}_m, t_{m+}) = 0 \quad \text{for } \Omega_3 \to \partial\Omega_{13}. \end{aligned} \tag{6.77}$$

From Luo (2005a), with $\mathbf{F}_{\alpha\beta}^{(0)} = (V, 0)^{\mathrm{T}}$ the grazing motion in Eq. (6.62) is guaranteed by the following conditions:

$$\begin{aligned} &\left[\mathbf{n}_{\partial\Omega_{\alpha\beta}}^{\mathrm{T}} \cdot \mathbf{F}^{(\alpha)}(\mathbf{x}_m, t_{m\pm})\right] = 0, \quad \alpha, \beta \in \{i, j\} \text{ and } \alpha \neq \beta; \\ &\left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathrm{D}\mathbf{F}^{(i)}(\mathbf{x}_m, t_{m\pm})\right] > 0, \quad \left[\mathbf{n}_{\partial\Omega_{ji}}^{\mathrm{T}} \cdot \mathrm{D}\mathbf{F}^{(j)}(\mathbf{x}_m, t_{m\pm})\right] < 0 \\ &\quad \text{for } \partial\Omega_{ij} \in \{\partial\Omega_{21}, \partial\Omega_{13}, \partial\Omega_{34}\} \end{aligned} \tag{6.78}$$

where

$$\mathrm{D}\mathbf{F}^{(i)}(\mathbf{x}, t) = \left(2F_i(\mathbf{x}, t), \nabla F_i(\mathbf{x}, t) \cdot \mathbf{F}^{(i)}(\mathbf{x}, t) + \frac{\partial F_i(\mathbf{x}, t)}{\partial t}\right)^{\mathrm{T}}. \tag{6.79}$$

By Eq. (6.73), we have

$$\begin{aligned} &\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}^{(i)}(\mathbf{x}, t) = F_i(\mathbf{x}, \Omega t), \\ &\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathrm{D}\mathbf{F}^{(i)}(\mathbf{x}, t) = \nabla F_i(\mathbf{x}, \Omega t) \cdot \mathbf{F}^{(i)}(\mathbf{x}, t) + \frac{\partial F_i(\mathbf{x}, \Omega t)}{\partial t}. \end{aligned} \tag{6.80}$$

From Eqs. (6.76) and (6.78), the force conditions for grazing motions are:

$$\begin{aligned} &F_\alpha(\mathbf{x}_m, \Omega t_m) = 0, \quad \alpha \in \{i, j\}; \\ &\nabla F_i(\mathbf{x}_m, \Omega t_{m\pm}) \cdot \mathbf{F}^{(i)}(\mathbf{x}_m, t_{m\pm}) + \frac{\partial F_i(\mathbf{x}_m, \Omega t_{m\pm})}{\partial t} > 0, \\ &\nabla F_j(\mathbf{x}_m, \Omega t_{m\pm}) \cdot \mathbf{F}^{(j)}(\mathbf{x}_m, t_{m\pm}) + \frac{\partial F_j(\mathbf{x}_m, \Omega t_{m\pm})}{\partial t} < 0 \\ &\quad \text{for } \partial\Omega_{ij} \in \{\partial\Omega_{21}, \partial\Omega_{13}, \partial\Omega_{34}\}. \end{aligned} \tag{6.81}$$

To illustrate the motion with flow barriers in nonsmooth dynamic systems, the basic mappings are introduced. With an initial condition (t_i, x_i, V), the direct integration of Eq. (6.56) yields

$$x = V(t - t_i) + x_i. \tag{6.82}$$

Substitution of Eq. (6.82) into (6.64) produces the forces for the very small-neighborhood of the stick motion ($\delta \to 0$) in the domains Ω_j ($j \in \{1, 3\}$). Because of the static friction jumping, the input and output forces at $(\mathbf{x}_m, t_m)$ on the boundary $\partial\Omega_{13}$ are:

$$F_{\text{in}}^{(j)}(\mathbf{x}_m, t_{m-}) = 2d_j V + c_j\left[V(t_m - t_i) + x_i\right] - A_0 \cos\Omega t_m + a_j^{(\text{in})}, \tag{6.83}$$

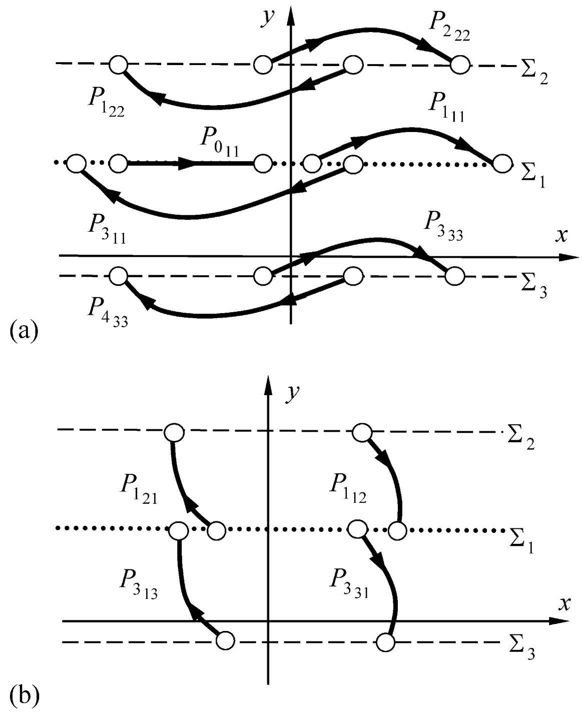


Figure 6.5. Regular and stick mappings: (a) local and stick mappings, (b) global mappings.

$$F_{\mathrm{b}}^{(j)}(\mathbf{x}_m, t_{m-}) = 2d_j V + c_j\big[V(t_m - t_i) + x_i\big] - A_0 \cos \Omega t_m + a_j^{(\mathrm{b})}, \quad (6.84)$$

where $a_1^{(\mathrm{in})} = -a_3^{(\mathrm{in})} = \mu_k F_N$ and $a_1^{(\mathrm{b})} = -a_3^{(\mathrm{b})} = \mu_s F_N$.

To develop generic mappings, the boundary should be numbered. The jumping discontinuous force boundary is represented by Σ_1, the other separation boundaries are Σ_2 and Σ_3. The three boundaries are

$$\Sigma_\alpha = \Sigma_\alpha^0 \cup \Sigma_\alpha^+ \cup \Sigma_\alpha^- \quad \text{for } \alpha = 1, 2, 3. \quad (6.85)$$

The corresponding, switching planes are defined as

$$\Sigma_\alpha^0 = \big\{(x_i, \Omega t_i) \mid \dot{x}_i(t_i) = V_\rho\big\}, \quad \Sigma_\alpha^\pm = \big\{(x_i, \Omega t_i) \mid \dot{x}_i(t_i) = V_\rho^\pm\big\} \quad (6.86)$$

where $V_\sigma^\pm = \lim_{\delta\to 0}(V_\sigma \pm \delta)$ for an arbitrarily small $\delta > 0$ and $\rho = 0, 1, 2$ for $\alpha = 1, 2, 3$. In phase plane, the trajectories in Ω_j starting and ending at the separation boundaries are sketched in Fig. 6.5. The starting and ending points for mappings $P_{j\beta\alpha}$ in Ω_j are $(x_i, \dot{x}_i, t_i)$ on Σ_α and $(x_{i+1}, \dot{x}_{i+1}, t_{i+1})$ on Σ_β, respectively. Notice that the indices $j = 1, 2, 3, 4$ and $\alpha, \beta = 1, 2, 3$ are for domains and boundaries. The stick mapping is $P_{0_{11}}$. Therefore, the mappings can be expressed through the switching sets as

$$\begin{aligned} &P_{1_{11}}:\Sigma_1^+ \xrightarrow{\Omega_1} \Sigma_1^+, \quad P_{3_{11}}:\Sigma_1^- \xrightarrow{\Omega_3} \Sigma_1^-,\\ &P_{2_{22}}:\Sigma_2^+ \xrightarrow{\Omega_2} \Sigma_2^+, \quad P_{1_{22}}:\Sigma_2^- \xrightarrow{\Omega_1} \Sigma_2^-,\\ &P_{4_{33}}:\Sigma_3^- \xrightarrow{\Omega_4} \Sigma_3^-, \quad P_{3_{33}}:\Sigma_3^+ \xrightarrow{\Omega_3} \Sigma_3^+ \end{aligned} \tag{6.87}$$

for the local mappings;

$$\begin{aligned} &P_{1_{21}}:\Sigma_1^+ \xrightarrow{\Omega_1} \Sigma_2^-, \quad P_{1_{12}}:\Sigma_2^- \xrightarrow{\Omega_1} \Sigma_1^+,\\ &P_{3_{31}}:\Sigma_1^- \xrightarrow{\Omega_3} \Sigma_3^+, \quad P_{3_{13}}:\Sigma_3^+ \xrightarrow{\Omega_3} \Sigma_1^- \end{aligned} \tag{6.88}$$

for the global mappings;

$$P_{0_{11}}:\Sigma_1^0 \xrightarrow{\partial\Omega_{13}} \Sigma_1^0 \tag{6.89}$$

for the stick mapping.

The governing equations for $P_{0_{11}}$ and $j \in \{1,3\}$ are

$$\begin{aligned} &-x_{i+1} + V(t_{i+1}-t_i) + x_i = 0,\\ &2d_j V + c_j\big[V(t_{i+1}-t_i) + x_i\big] - A_0\cos\Omega t + a_j^{(b)} = 0. \end{aligned} \tag{6.90}$$

For the nonstick motion, the governing equations of mapping $P_{j_{\beta\alpha}}$ ($j = 1,2,3,4$ and $\alpha,\beta = 1,2,3$) are:

$$\begin{aligned} &f_1^{(j_{\beta\alpha})}(x_i,\Omega t_i, x_{i+1},\Omega t_{i+1}) = 0,\\ &f_2^{(j_{\beta\alpha})}(x_i,\Omega t_i, x_{i+1},\Omega t_{i+1}) = 0. \end{aligned} \tag{6.91}$$

From the above mappings, the periodic motions for such a periodically forced, frictional oscillator can be referred to Luo and Zwiegart (2005).

For numerical simulations, the parameters $m = 5$, $d_{1,2} = 0.1$, $c_{1,2} = 30$, $V = 3$, $V_1 = 4.5$, $V_2 = 1.5$, $\mu_s = 0.5$, $\mu_k = 0.4$, $\mu_{1,3} = 0.1$, $\mu_{2,4} = 0.5$ and $g = 9.8$ are used. Consider a nonstick periodic motion relative to $P_{3_{33}} \circ P_{4_{33}}$ for illustration. This periodic motion does not have the intersection with the boundary $\partial\Omega_{13}$. The periodic motion switches at the boundary $\partial\Omega_{34}$. Because the friction is continuous with the piecewise linearity as in Eq. (6.54), the total force on the boundary $\partial\Omega_{34}$ is continuous, but the derivative of the force is discontinuous. The phase plane, force distributions along the displacement and velocity, and the responses of displacement, velocity and acceleration for the periodic motion relative to mapping $P_{4_{33}3_{33}}$ are plotted respectively in Figs. 6.6(a)–(f) for $\Omega = 5$ and

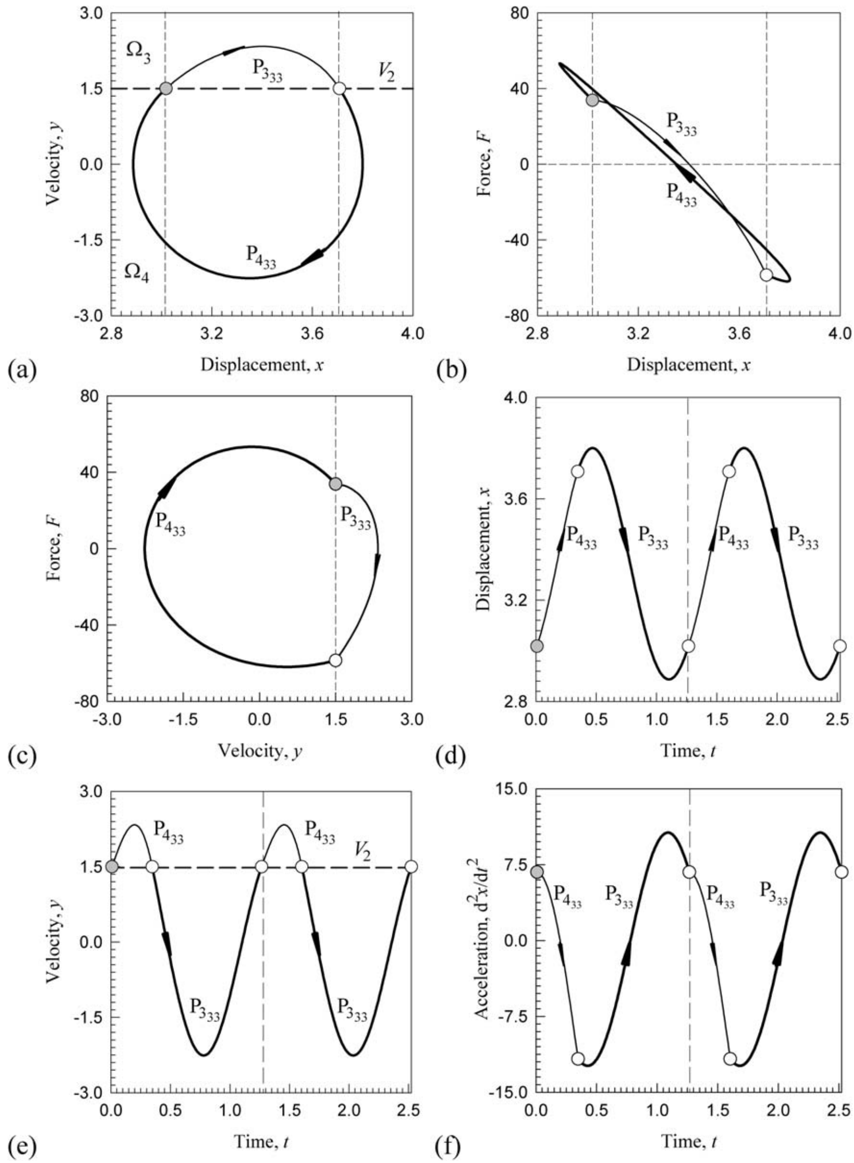


Figure 6.6. Periodic responses of mapping $P_{4_{33}} \circ P_{3_{33}}$: (a) phase plane, (b) force distribution along displacement, (c) force distribution along velocity, (d) displacement, (e) velocity, and (f) acceleration for $\Omega = 5$ and $Q_0 = 70$ with $(\Omega t_i, x_i, \dot{x}_i) \approx (0.0458, 3.0183, 1.50)$.

$Q_0 = 70$ with $(\Omega t_i, x_i, \dot{x}_i) \approx (0.0458, 3.0183, 1.50)$. The responses in Ω_3 and Ω_4 are represented by the thin and dark curves, accordingly. The circular symbols are switching points, and the gray filled symbol is the starting point of the periodic motion. The arrow is the direction of the periodic motion. The phase plane is clearly presented in Fig. 6.6(a). In Figs. 6.6(b) and (c), the force is not smooth at the boundary $\partial\Omega_{34}$, and the corresponding acceleration in Fig. 6.6(f) is not smooth as well. The displacement and velocity responses in Figs. 6.6(d)–(e) are very smooth. This motion is not interacted the boundary $\partial\Omega_{13}$. In addition, the corresponding mappings are labeled in all the plots. Once the starting is changed to the another switching point, the mapping structure of the periodic motion becomes $P_{3_{33}4_{33}} = P_{3_{33}} \circ P_{4_{33}}$. Therefore, the two mapping structures present the same periodic motion except for the different initial conditions.

Consider a stick periodic motion relative to mapping $P_{3_{31}0_{11}3_{13}4_{33}}$. The excitation frequency and amplitude $\Omega = 1$ and $Q_0 = 70$ with the initial conditions $(\Omega t_i, x_i, \dot{x}_i) \approx (0.6672, 6.5814, 1.50)$ are used, and the other parameters are the same as in the first example. Again, the phase plane, force distributions along displacement and velocity, and displacement, velocity and acceleration responses for the periodic motion of $P_{3_{31}0_{11}3_{13}4_{33}}$ are shown in Figs. 6.7(a)–(f), respectively. For this periodic motion, the sliding motion along the boundary $\partial\Omega_{13}$ exists. Since the friction force discontinuity on $\partial\Omega_{13}$ exists in Eq. (6.54), the discontinuity of the total force can be observed on both sides of the neighborhood of the boundary. Because the static and kinetic fraction forces are different, the flow barriers exist. Therefore, the output force relative to the flow barriers are defined in Eq. (6.84). $F_{\mathrm{b}}^{(\alpha)} \triangleq F_{\mathrm{b}}^{(\alpha)}(\mathbf{x}_m, t_{m-})$ is depicted to observe the force criteria for the disappearance of the stick motion. The stick motion is a straight line along the discontinuous boundary in phase plane in Fig. 6.7(a). The force nonsmoothness on the boundary $\partial\Omega_{34}$ between the two domains Ω_3 and Ω_4 is clearly observed in Fig. 6.7(b). The input forces are continuous in the corresponding neighborhoods of the two domains, thus, the input force is not presented. In Fig. 6.7(b), the stick motion disappears at $F_{\mathrm{b}}^{(3)} = 0$, which satisfies the conditions in Eq. (6.76), and it indicates that the nonfriction force overcomes the static friction force (i.e., flow barriers). Furthermore, the oscillator will oscillate on the moving belt, and the kinetic friction force is used for motion in domain Ω_3. So the corresponding force from zero jumps to the negative one (i.e., $F_3(\mathbf{x}, t) < 0$), as illustrated in Fig. 6.7(b). Since the input force $F_{\mathrm{in}}^{(3)} > 0$ and the output force $F_{\mathrm{in}}^{(1)} < 0$, the stick motion appears. The nonsmoothness of the forces on the boundary $\partial\Omega_{34}$ is also observed. Such force characteristics of stick motion and domain switching are presented in Fig. 6.7(c) as well. From Figs. 6.7(d) and (e), the displacement and velocity are continuous. However, the acceleration is discontinuous at $\partial\Omega_{13}$ and nonsmooth at $\partial\Omega_{34}$ because of the force discontinuity, which is observed in Fig. 6.7(f).

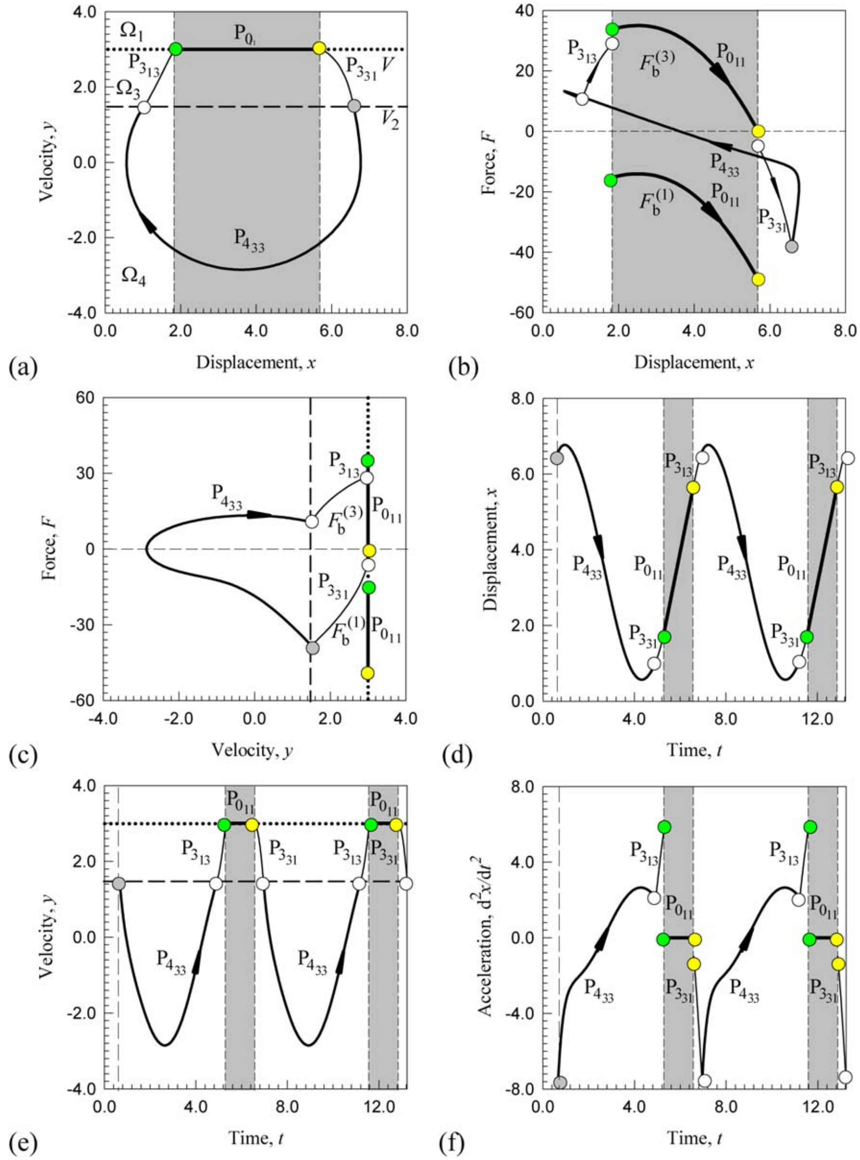


Figure 6.7. Periodic responses of mapping $P_{3_{31}} \circ P_{0_{11}} \circ P_{3_{13}} \circ P_{4_{33}}$: (a) phase plane, (b) force distribution along displacement, (c) force distribution along velocity, (d) displacement, (e) velocity, and (f) acceleration for $\Omega = 1$ and $Q_0 = 70$ with $(\Omega t_i, x_i, \dot{x}_i) \approx (0.6672, 6.5814, 1.50)$.

Chapter 7

Transport Laws and Mapping Dynamics

Once a real flow arrives to the discontinuous boundary, the real flow cannot pass through the boundary, but the flow may slide on the boundary. In Chapter 6, if the flow barrier on the separation boundary is permanent or semi-permanent one, the flow never passes over the barrier into another accessible domain. Therefore, a suitable transport law is required to overcome the permanent flow barrier. In addition, if the accessible domain is separated by the nonaccessible domain, a flow from one accessible domain to another accessible one may require a transport law to continue the flow in the global system. In this chapter, the transport law will be introduced as a specific map which transports the flow from an accessible domain to another accessible domain. Once a specific transport law between the two accessible domains is determined, the global flow in all the possible accessible domains can continue. Further, the mapping dynamics of the global flow will be discussed. The mapping structure will be introduced for determining complicated periodic flows. A linear impact dynamical system will be presented for demonstration of the methodology to determine the global periodic flow in discontinuous dynamical systems.

7.1. Classification of discontinuity

Consider a flow $\mathbf{\Phi}(\mathbf{x}, t)$ of Eq. (2.1) to be C^1-discontinuous (or C^0-continuous) on a boundary curve $\partial\Omega_{ij}$ (or hyper-surface), as shown in Fig. 7.1. For the C^1-discontinuous flow $\mathbf{\Phi}(\mathbf{x}, t) \equiv \mathbf{\Phi}(\mathbf{x}, \mathbf{x}_0, t, t_0)$, the points $\mathbf{x}^{(i)}$ and $\mathbf{x}^{(j)}$ relative respectively to the domains Ω_i and Ω_j on the boundary $\partial\Omega_{ij}$ are identical (i.e., $\mathbf{x}^{(i)} = \mathbf{x}^{(j)}$) for the same time t_m, but the flow gradients $\nabla\mathbf{\Phi}^{(i)}(\mathbf{x}^{(i)}, t_{m\pm})$ and $\nabla\mathbf{\Phi}^{(j)}(\mathbf{x}^{(j)}, t_{m\mp})$ on the boundary $\partial\Omega_{ij}$ are different (i.e., $\nabla\mathbf{\Phi}^{(i)}(\mathbf{x}^{(i)}, t_{m\pm}) \neq \nabla\mathbf{\Phi}^{(j)}(\mathbf{x}^{(j)}, t_{m\mp})$). In the domain Ω_i, there is a flow $\mathbf{\Phi}^{(i)}(\mathbf{x}, t)$ from the starting point p_{m-1} with coordinates $\mathbf{x}^{(i)}_{(m-1)+} \equiv \mathbf{x}^{(i)}(t_{(m-1)+})$ to the ending point p_m with $\mathbf{x}^{(i)}_{m+}$. In the domain Ω_j, there is a flow $\mathbf{\Phi}^{(j)}(\mathbf{x}, t)$ from the starting point p_m with coordinates $\mathbf{x}^{(j)}_{m-}$ to the ending point p_{m+1} with $\mathbf{x}^{(j)}_{(m+1)-}$. Thus, the flow $\mathbf{x}^{(i)} \cup \mathbf{x}^{(j)}$ is C^1-discontinuous on the boundary $\partial\Omega_{ij}$. Such a boundary $\partial\Omega_{ij}$ is termed the

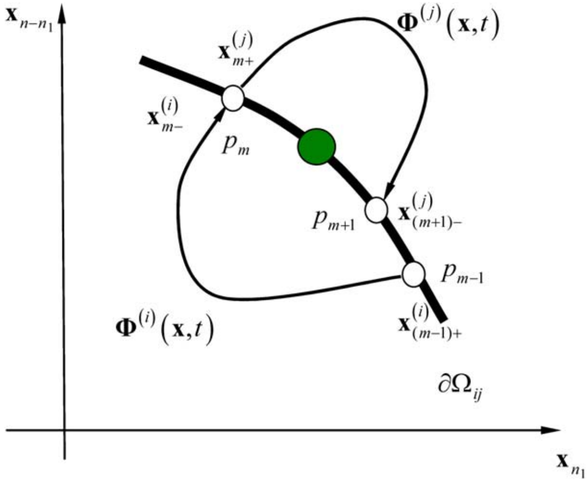


Figure 7.1. The C^1-discontinuous flow $\mathbf{\Phi}(\mathbf{x}, t)$ on the boundary $\partial\Omega_{ij}$ (or hyper-surface). The largest circle symbol is the connecting set.

C^1-discontinuous one for such a discontinuous system. The C^1-dynamical system is also called the nonsmooth dynamical system in mechanical engineering because the trajectory is continuous. The mathematical definition is given as follows:

DEFINITION 7.1. For a discontinuous system in Eq. (2.1), once the flow $\mathbf{x}^{(\alpha)}(t)$ ($\alpha \in \{i, j\}$) arrives to the boundary $\mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m, the boundary $\partial\Omega_{ij}$ is termed the first-order derivative discontinuous (or C^1-discontinuous) boundary of the flow $\mathbf{x}^{(\alpha)}(t) \cup \mathbf{x}^{(\beta)}(t)$ ($\beta \in \{i, j\}, \alpha \neq \beta$) in Eq. (2.1) if

$$\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(\beta)}(t_{m+}), \tag{7.1}$$

$$\nabla\mathbf{\Phi}^{(\alpha)}\big(\mathbf{x}^{(\alpha)}, t_{m-}\big) \neq \nabla\mathbf{\Phi}^{(\beta)}\big(\mathbf{x}^{(\beta)}, t_{m+}\big). \tag{7.2}$$

The singularity and dynamics on discontinuous dynamical systems with such a discontinuous boundary has been discussed in Chapters 2–6. Even if the permanent flow barrier exists, the flow may slide on the boundary, but the flow cannot pass through the boundary and into the domain with the permanent flow barrier. For this case, usually, no transport law on the boundary exists. For special purposes, the transport laws can be added on such a discontinuous boundary, which will be discussed later. From Chapter 6, once the input barrier $\mathbf{f}^{(\alpha)}_{\alpha\beta}(\mathbf{x}_m, t_{m\pm})$ is very large (i.e., $\mathbf{f}^{(\alpha)}_{\alpha\beta}(\mathbf{x}_m, t_{m\pm}) \to \infty$), the vector field $\mathbf{f}^{(\alpha)}(\mathbf{x}_m, t_{m\pm})$ cannot overcome the input barrier $\mathbf{f}^{(\alpha)}_{\alpha\beta}(\mathbf{x}_m, t_{m\pm})$ to get on the boundary $\partial\Omega_{ij}$. The incoming flow $\mathbf{x}^{(\alpha)}(t)$ of Eq. (2.1) to the boundary $\partial\Omega_{ij}$ can return back into the domain Ω_α

until the vector field changes the direction or the incoming flow can be transported to the different domain by a certain transport law. The mathematical description is given as follows:

DEFINITION 7.2. For an incoming flow $\mathbf{x}^{(\alpha)}(t)$ ($\alpha \in \{i, j\}$) in Eq. (2.1) approaching to the boundary $\mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m, there is an input flow barrier $\mathbf{f}^{(\alpha)}_{\alpha\beta}(\mathbf{x}_m, t_{m\pm})$ ($\beta \in \{i, j\}, \alpha \neq \beta$) on the α-side of the boundary $\partial\Omega_{ij}$. The boundary $\partial\Omega_{ij}$ to the incoming flow $\mathbf{x}^{(\alpha)}(t)$ is permanently-nonpassable if the vector field cannot overcome the flow barrier, i.e.,

$$\left\|\mathbf{F}^{(\alpha)}\right\| < \left\|\mathbf{f}^{(\alpha)}_{\alpha\beta}\right\| \quad \text{for all } \mathbf{x}_m \in \partial\Omega_{ij} \text{ and } t_m. \tag{7.3}$$

To extend the Definition 7.2, the definition of the forbidden boundary is presented. Once the forbidden boundary exists, the transport law must exist for the motion from one domain to another one.

DEFINITION 7.3. For two incoming flow $\mathbf{x}^{(\alpha)}(t)$ ($\alpha \in \{i, j\}$) in Eq. (2.1) approaching to the boundary $\mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m, there are two input flow barriers $\mathbf{f}^{(\alpha)}_{\alpha\beta}(\mathbf{x}_m, t_{m\pm})$ ($\beta \in \{i, j\}, \alpha \neq \beta$) on the both sides of the boundary $\partial\Omega_{ij}$. The boundary $\partial\Omega_{ij}$ for the flow $\mathbf{x}^{(\alpha)}(t) \cup \mathbf{x}^{(\beta)}(t)$ is forbidden if the boundary $\partial\Omega_{ij}$ to both of the incoming flows $\mathbf{x}^{(\alpha)}(t)$ ($\alpha \in \{i, j\}$) is permanently-nonpassable.

Consider a separation boundary $\partial\Omega_{ij}$ to be permanently-nonpassable. In a domain Ω_α ($\alpha \in \{i, j\}$), the discontinuous boundary set is defined by $\partial\Omega^{(\alpha)}_{ij}$. The discontinuous boundary set $\partial\Omega^{(\alpha)-}_{ij}$ represents just before the flow arrives to the boundary, and $\partial\Omega^{(\alpha)+}_{ij}$ just after the flow leaves the boundary. The connecting set for the two discontinuous boundary sets is denoted by $\Gamma^{(\alpha)}_{ij}$. Therefore, a α-side boundary $\partial\Omega^{(\alpha)}_{ij} = \partial\Omega^{(\alpha)-}_{ij} \cup \partial\Omega^{(\alpha)+}_{ij} \cup \Gamma^{(\alpha)}_{ij}$ exists. For any permanently-nonpassable boundary, there is at least a transport law to map the flow passing over the connecting set or to another domain. However, the incoming flow $\mathbf{x}_\alpha$ to the boundary $\partial\Omega^{(\alpha)}_{ij}$ will stay on the boundary until the vector field $\mathbf{F}^{(\alpha)}(\mathbf{x}_m, t_m)$ changes its direction with varying t_m. The hyperbolic and parabolic phase portraits in vicinity of a forbidden boundary are depicted in Fig. 7.2.

The forbidden boundary is expressed through a forbidden zone. In fact, the width of the fictional forbidden zone is zero. To cross over the forbidden zone, a transport law should be exerted. If points $\mathbf{x}^{(i)}$ and $\mathbf{x}^{(j)}$ on the boundary for the same time t_m are different (i.e., $\mathbf{\Phi}^{(i)}(\mathbf{x}^{(i)}_m, t_m) \neq \mathbf{\Phi}^{(j)}(\mathbf{x}^{(j)}_m, t_m)$), the discontinuous system is C^0-discontinuous on the boundary. The relationship is determined through at least a certain transport law, which will be discussed in the next section. The mathematical definition for such a discontinuous boundary is given as follows:

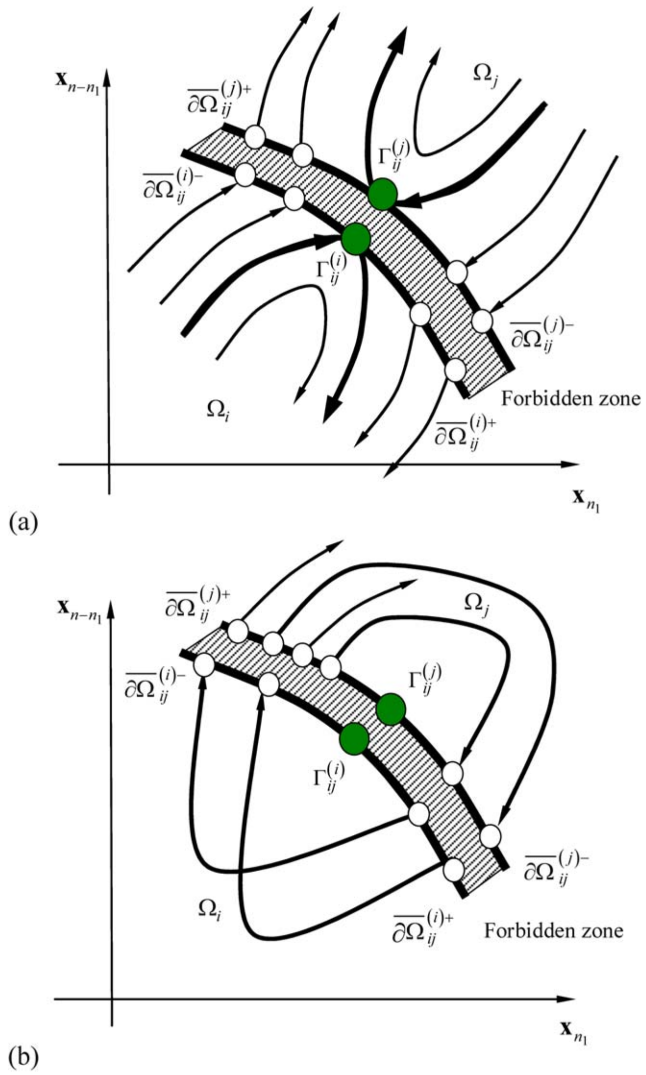


Figure 7.2. (a) Hyperbolic and (b) parabolic phase portraits in vicinity of the forbidden discontinuous boundary $\partial\Omega_{ij}$ without a discontinuous law. The largest, solid circular circle is the connecting set $\Gamma_{ij}^{(\alpha)}$ ($\alpha \in \{i, j\}$). The largest solid curves with circular symbols are the discontinuous boundary set. The forbidden zone is fictitious.

DEFINITION 7.4. For a discontinuous system in Eq. (2.1), once the flow $\mathbf{x}^{(\alpha)}(t)$ ($\alpha \in \{i, j\}$) arrives to the boundary $\mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m, the boundary $\partial\Omega_{ij}$ is

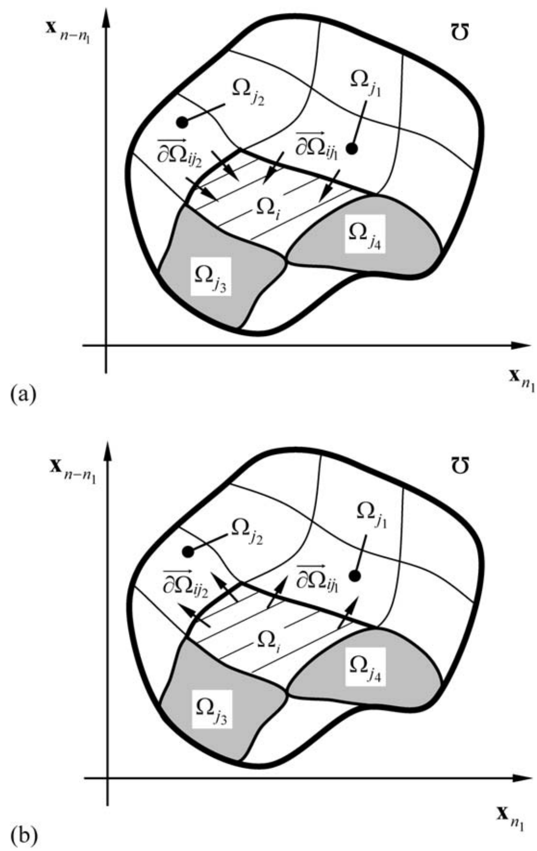


Figure 7.3. (a) Sink and (b) source domains Ω_i. The white subdomains are accessible. The gray subdomains are inaccessible domain. The sink and source domain is shaded.

termed the *zero-order derivative discontinuous* (or C^0-discontinuous) boundary of the flow $\mathbf{x}^{(\alpha)}(t) \cup \mathbf{x}^{(\beta)}(t)$ ($\beta \in \{i, j\}, \alpha \neq \beta$) in Eq. (2.1) if for time t_m

$$\mathbf{x}^{(\alpha)}(t_{m-}) \in \partial\Omega_{ij} \quad \text{and} \quad \mathbf{x}^{(\beta)}(t_{m+}) \in \partial\Omega_{ij}, \tag{7.4}$$

$$\mathbf{\Phi}^{(\alpha)}\big(\mathbf{x}^{(\alpha)}, t_{m-}\big) \neq \mathbf{\Phi}^{(\beta)}\big(\mathbf{x}^{(\beta)}, t_{m+}\big). \tag{7.5}$$

To explain the above two concepts geometrically in phase space, the sink and source domains are sketched through the shaded area in Fig. 7.3. The neighbored subdomains include the accessible and inaccessible subdomains, which are depicted through the white and gray colors, respectively. The boundaries between the accessible and inaccessible domain are permanently-nonpassable. For the sink domain, the flows to its passable boundaries are towards the inside of the domain, as shown in Fig. 7.3(a). However, for the source domain, the flows to its boundaries are towards the outside of domain, as shown in Fig. 7.3(b). If all the neighbored subdomains of the accessible domain Ω_i are inaccessible, the corresponding boundaries of Ω_i should be permanently-nonpassable. If flows in domain Ω_i pass over the boundary and into the other nonadjacent accessible domains, at least one transport law should be exerted.

If all the neighbored subdomains of the accessible domain Ω_i are accessible and the corresponding boundaries of Ω_i are semi-passable only in one direction, then the accessible domain Ω_i is sink or source. Consider a closed accessible domain Ω_i as an example for illustration. The sink and source domains are sketched in Figs. 7.4(a) and (b) for a better understanding of the above definitions. The solid closed surface is the boundary of the closed domain Ω_i. The arrows are the flow going in the sink domain and out the source domain, respectively. This concept already extended the source and sink of equilibrium in continuous dynamical systems. For the inside of the two domains, there is a continuous dynamic system. In mechanical systems, there are two kinds of semi-passable boundary at the same time to make a flow get into and out the domain.

DEFINITION 7.5. The accessible domain $\Omega_\alpha \subset \Omega$ for discontinuous dynamical systems in Eq. (2.1) is termed a *sink* domain if for all the corresponding boundaries of Ω_α with all the neighbored domains Ω_β, except for permanently-nonpassable boundaries, the rest of the boundaries of Ω_α must be semi-passable $\overrightarrow{\partial\Omega}_{\beta\alpha}$.

DEFINITION 7.6. The accessible domain $\Omega_\alpha \subset \Omega$ for discontinuous dynamical systems in Eq. (2.1) is termed a *source* domain if for all the corresponding boundaries of Ω_α with all neighbored domains Ω_β, except for permanently-nonpassable boundaries, the rest of the boundaries of Ω_α must be semi-passable $\overrightarrow{\partial\Omega}_{\alpha\beta}$.

7.2. Transport laws

Before discussing transport laws on the discontinuous boundary $\partial\Omega_{ij}$, the *input* and *output* boundaries are first introduced herein.

DEFINITION 7.7. For a discontinuous system in Eq. (2.1), the discontinuous boundary $\partial\Omega_{ij}$ to which only the flow in the subdomain Ω_α ($\alpha \in \{i, j\}$) arrives is *termed* the α-side *input* boundary $\partial\Omega^{\downarrow\alpha}_{\alpha\beta}$ ($\beta \in \{i, j\}$ and $\alpha \neq \beta$).

DEFINITION 7.8. For a discontinuous system in Eq. (2.1), the discontinuous boundary $\partial\Omega_{ij}$ on which only a flow leaves for the subdomain Ω_α ($\alpha \in \{i, j\}$) is *termed* the α-side *output* boundary $\partial\Omega^{\uparrow\alpha}_{\beta\alpha}$ ($\beta \in \{i, j\}$ and $\alpha \neq \beta$).

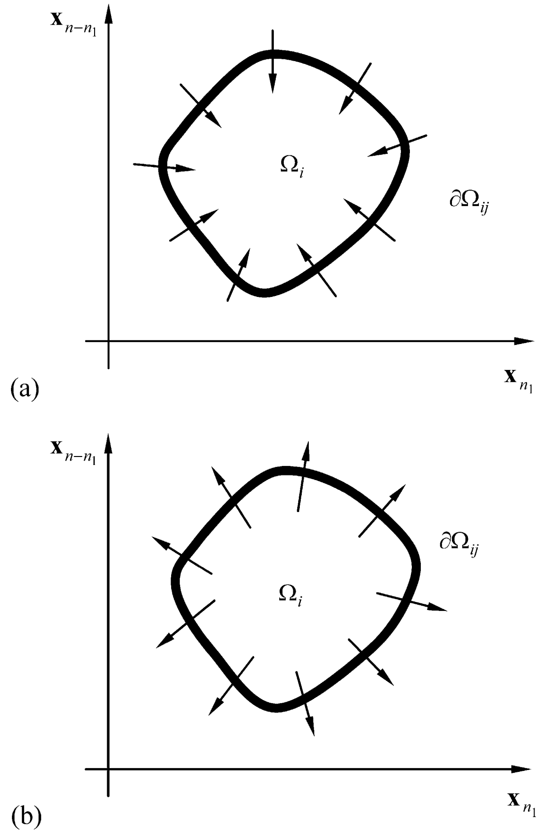


Figure 7.4. Two closed accessible domains: (a) sink and (b) source. The solid closed curve is the boundary of the closed accessible domain Ω_i. The arrows are the flow going into the sink domain and going out the source domains, respectively.

From the above two concepts, whatever the boundary is semi-passable (i.e., $\overrightarrow{\partial\Omega}_{ij}$) or nonpassable (i.e., $\overline{\partial\Omega}_{ij}$), only if the flow in the corresponding domain can come to the boundary, this boundary is defined the input boundary. On the other hand, only if the flow can leave this boundary into the corresponding domain, the boundary is called the output boundary.

To restrict our discussion, the investigation focuses on making the flow continuous on the C^0-discontinuous boundaries. Consider a C^0-discontinuous boundary to be the part of the boundary for the unbounded, universal domain first. Since the universal domain is unbounded, the boundary is located at the infinity. For such a case, no transport rule or law is required. Hence, a more strict description is given:

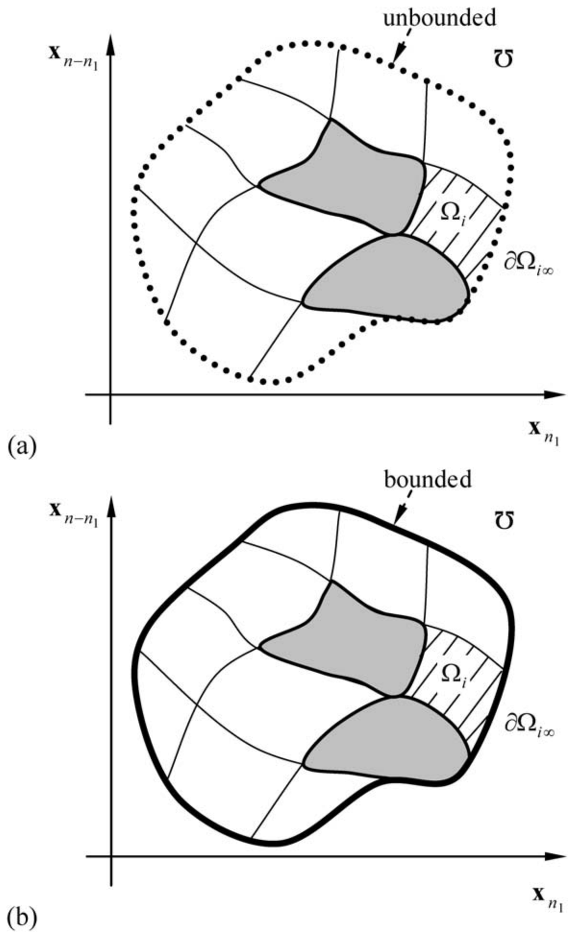


Figure 7.5. (a) Unbounded and (b) bounded domains Ω_i. The white subdomains are accessible. The gray subdomains are inaccessible domain. The shaded domain is a specified accessible domain.

DEFINITION 7.9. Under Assumptions (A1)–(A3), if the discontinuous dynamic system of Eq. (2.1) in an unbounded domain Ω_i possesses a bounded flow, there are no any discontinuous laws which can be exerted on the boundary $\partial\Omega_{i\infty} = \Omega_i \cap \mho \neq \varnothing$.

To show the above definition intuitively, the boundaries for unbounded and bounded, universal domains are sketched in Fig. 7.5 through the dotted and solid curves, respectively. The white subdomains are accessible. The gray subdomains are inaccessible. The shaded domain is a specified accessible domain.

From Assumptions (A2)–(A3) in Chapter 2, the flow in the unbounded subdomain Ω_i must be bounded because of $\partial\Omega_{i\infty} = \Omega_i \cap \mho \neq \varnothing$ to make physical flows in the entire discontinuous system exist. For a bounded, universal domain $\mho$, there is a common boundary $\partial\Omega_{i\infty} = \Omega_i \cap \mho \neq \varnothing$ between the domain $\mho$ and Ω_i is nonpassable. If the common boundary $\partial\Omega_{i\infty}$ possesses at least a pair of the input and output sub-boundaries on the i-side of $\partial\Omega_{i\infty}$ (i.e., $\partial\Omega^{\downarrow i}_{i\infty}$ and $\partial\Omega^{\uparrow i}_{i\infty}$). On the nonpassable boundary, there is at least a transport law to map the input discontinuous boundary into the output discontinuous boundary. If the boundary $\partial\Omega_{i\infty}$ is the input or output boundary, a transport law on the boundary must exist to map $\partial\Omega_{i\infty}$ as an *input* boundary into an output boundary or to map an input boundary into $\partial\Omega_{i\infty}$ as an output boundary. Otherwise, the input boundary becomes an *attractive* boundary set (i.e., "blackhole"), or the output boundary becomes *source* to repel the flow.

For the boundary $\partial\Omega_{i\infty}$, the *input* and *output* boundaries of flow are represented by $\partial\Omega^{\downarrow i}_{i\infty}$ and $\partial\Omega^{\uparrow i}_{i\infty}$, respectively. A singular set $\Gamma^{(i)}_{i\infty}$ connects the *input* and *output* boundaries. The transport law between the input and output boundaries is given through the following definition.

DEFINITION 7.10. For a discontinuous system in Eq. (2.1), under Assumptions (A1)–(A3), there is a flow $\mathbf{x}^{(i)} \in \Omega_i$ with a common boundary

$$\partial\Omega_{i\infty} = \bigcup_{j=1}^{j_i} {}^{(j)}\partial\Omega^{\downarrow i}_{i\infty} \cup \bigcup_{k=1}^{k_i} {}^{(k)}\partial\Omega^{\uparrow i}_{i\infty} \cup \bigcup_{n=1}^{n_i} {}^{(n)}\Gamma^{(i)}_{i\infty} \tag{7.6}$$

where $\{j_i, k_i, n_i\}$ are the maximum numbers of the input and output boundaries and connecting sets. A map ${}^{(jk)}T_{i\infty}$ transporting a point ${}^{(j)}\mathbf{x}^{(i)}_{m-} \in {}^{(j)}\partial\Omega^{\downarrow i}_{i\infty}$ to a point ${}^{(k)}\mathbf{x}^{(i)}_{m+} \in {}^{(k)}\overleftarrow{\partial\Omega}^{\uparrow i}_{i\infty}$ for time t_m without time consumption is termed *the transport law* on the common boundary $\partial\Omega_{i\infty}$, i.e.,

$${}^{(jk)}T_{i\infty} : {}^{(j)}\mathbf{x}^{(i)}_{m-} \to {}^{(k)}\mathbf{x}^{(i)}_{m+}, \tag{7.7}$$

which is governed by the $(n-1)$-dimensional one-to-one vector function $\mathbf{g}_{i\infty} \in \Re^{n-1}$ as

$$\mathbf{g}_{i\infty}\big({}^{(j)}\mathbf{x}^{(i)}_{m-}, {}^{(k)}\mathbf{x}^{(i)}_{m+}, \mathbf{p}^{(i\infty)}\big) = 0 \quad \text{on } \partial\Omega_{i\infty} \tag{7.8}$$

with a parameter vector $\mathbf{p}^{(i\infty)} \in \Re^{k_i}$ relative to the boundary $\partial\Omega_{i\infty}$.

The transport law on the common boundary $\partial\Omega_{i\infty}$ can be explained geometrically in Fig. 7.6 through the dashed curve arrow. The white circular points are for transporting from the input onto output boundaries of the common boundary. The transport law can be given to discontinuous dynamical systems, or can generically exist in discontinuous dynamical systems. For example, the ball mo-

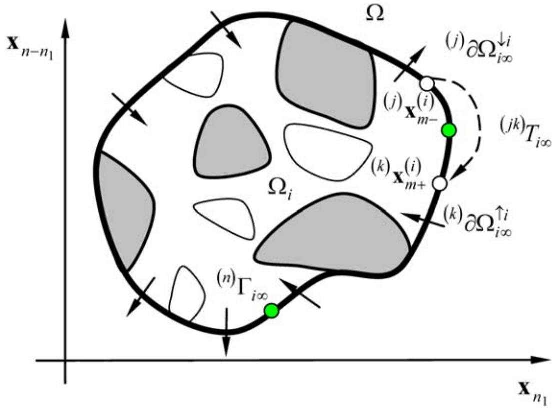


Figure 7.6. Transport law on the common boundary $\partial\Omega_{i\infty}$. The white and gray areas are accessible and inaccessible domains, respectively. The arrows represents the flow input and output sub-boundaries on $\partial\Omega_{i\infty}$. The dark circular symbols are connecting sets, and the white circular points are for flow transporting.

tion in an impact damper is discontinuous. The boundary of the impact damper is discontinuous. The impact law is a transport law to map the motion on the input boundary onto the output boundary. The ball motion in the impact damper can be referred to Masri and Caughey (1966), Masri (1970) and Han et al. (1995).

Consider $^{(j)}\mathbf{x}_{m-}^{(i)} \in {}^{(j)}\partial\Omega_{i\infty}^{\downarrow i} \subset \Re^{n-1}$ on the i-side, input boundary, which is satisfy $^{(j)}\varphi_{i\infty}(^{(j)}\mathbf{x}_{m-}^{(i)}, t_{m-}) = 0$. Because the function for the separation boundary is unique, one of the n-components in $^{(j)}\mathbf{x}_{m-}^{(i)}$ can be expressed by the $(n-1)$-components. Similarly, for $^{(k)}\mathbf{x}_{m+}^{(i)} \in {}^{(k)}\partial\Omega_{i\infty}^{\uparrow i} \subset \Re^{n-1}$ on the i-side, output boundary, the boundary function $^{(k)}\varphi_{i\infty}(^{(k)}\mathbf{x}_{m+}^{(i)}, t_{m+}) = 0$ should be satisfied. One of the n-components in $^{(k)}\mathbf{x}_{m+}^{(i)}$ can also expressed by the other $(n-1)$-components. Therefore, the transport law for mapping point $^{(j)}\mathbf{x}_{m-}^{(i)}$ to $^{(k)}\mathbf{x}_{m+}^{(i)}$ should be a one-to-one, $(n-1)$-dimensional vector function. It indicates the flow through this transport law is one-to-one. In Definition 7.10, the transport law is defined on the same domain with the input and output boundaries. The transport law between the boundaries of two separated, accessible subdomains is defined in the following way:

DEFINITION 7.11. For a discontinuous system in Eq. (2.1), under Assumptions (A1)–(A3), two accessible domains Ω_α ($\alpha \in \{i, j\}$) exist. For two flows $\mathbf{x}^{(\alpha)} \in \Omega_\alpha$, there is a map T_{ij}^{∞} transporting a point $\mathbf{x}_{m-}^{(i)} \in \partial\Omega_{i\infty}^{\downarrow i}$ to the point $\mathbf{x}_{m+}^{(j)} \in \partial\Omega_{j\infty}^{\uparrow j}$ for time t_m without time consumption is termed *the transport law* from

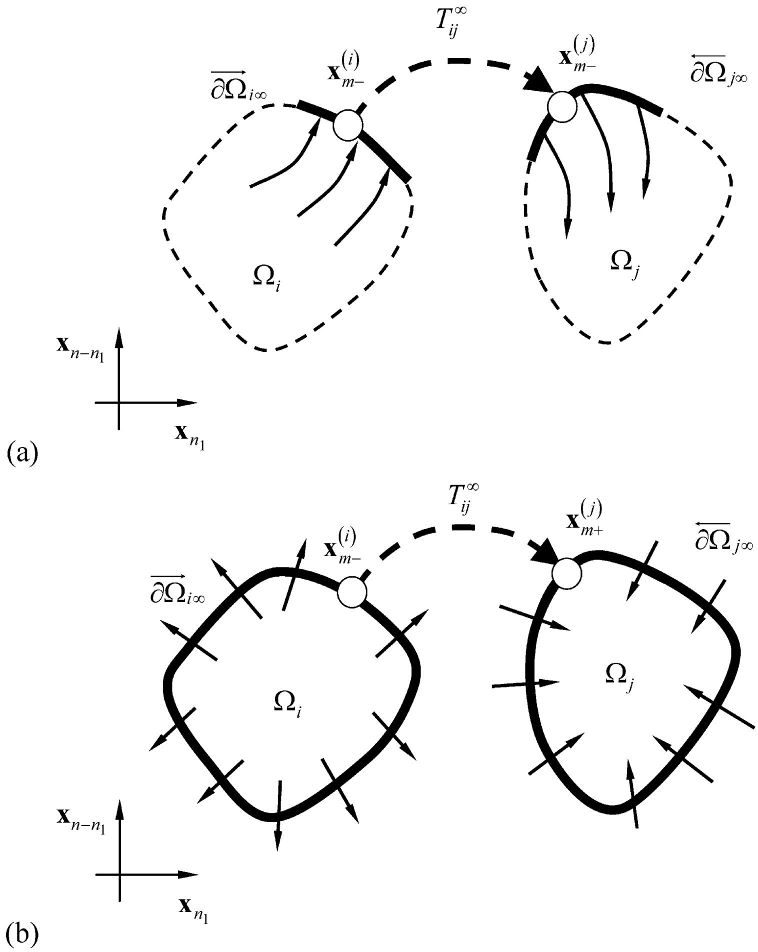


Figure 7.7. The transport laws mapping from (a) the output to input boundaries to the two separated domains, and (b) the source to sink domains on the common boundary $\partial\Omega_{\alpha\infty}$ ($\alpha \in \{i, j\}$).

$\partial\Omega_{i\infty}^{\uparrow i}$ to $\partial\Omega_{j\infty}^{\uparrow j}$, i.e.,

$$T_{ij}^{\infty} : \mathbf{x}_{m-}^{(i)} \to \mathbf{x}_{m+}^{(j)}, \tag{7.9}$$

which is governed by the $(n-1)$-dimensional one-to-one vector function $\mathbf{g}_{ij}^{\infty} \in \Re^{n-1}$ as

$$\mathbf{g}_{ij}^{\infty}\left(\mathbf{x}_{m-}^{(i)}, \mathbf{x}_{m+}^{(j)}, \mathbf{p}_{ij}^{\infty}\right) = 0 \quad \text{from } \partial\Omega_{i\infty}^{\downarrow i} \text{ to } \partial\Omega_{j\infty}^{\uparrow j} \tag{7.10}$$

with a parameter vector $\mathbf{p}_{ij}^{\infty} \in \Re^{k_{ij}^{\infty}}$.

This transport law is noninvertible. However, the transport law from the output boundary to the input boundary of two separated, accessible, subdomains is presented in Fig. 7.7. If the output and input boundaries of two separated domains are closed surfaces, the transport law will transport the flow from the *source* domain into the *sink* one. The transport law implies that the flows between two accessible, sub-systems can be exchanged. The above discussion about the transport law is relative to the common boundary between the subdomain and the universal domain. The further discussion is about the transport law on the regular boundary between two accessible subdomains. The transport law on the boundary between the two accessible subdomains will be discussed first, and then the generalized transport law on the boundaries between the accessible and/or inaccessible domains will be discussed.

DEFINITION 7.12. For a discontinuous system in Eq. (2.1), under Assumptions (A1)–(A3), two adjacent accessible domains Ω_α ($\alpha \in \{i, j\}$) exist. A map $T_{i\beta}$ transporting $\mathbf{x}_{m-}^{(i)} \in \partial\Omega_{ij}^{\downarrow i}$ to $\mathbf{x}_{m+}^{(\beta)} \in \partial\Omega_{\alpha\beta}^{\uparrow\beta}$ ($\beta \in \{i, j\}$ and $\alpha \neq \beta$) with $\mathbf{x}_{m-}^{(i)} \neq \mathbf{x}_{m+}^{(\beta)}$ is termed *a transport law* from $\partial\Omega_{ij}^{\downarrow i}$ to $\partial\Omega_{\alpha\beta}^{\uparrow\beta}$, i.e.,

$$T_{i\beta} : \mathbf{x}_{m-}^{(i)} \to \mathbf{x}_{m+}^{(\beta)} \tag{7.11}$$

which is governed by the $(n-1)$-dimensional one-to-one vector function $\mathbf{g}_{i\beta} \in \Re^{n-1}$ as

$$g_{i\beta}\left(\mathbf{x}_{m-}^{(i)}, \mathbf{x}_{m+}^{(\beta)}, \mathbf{p}_{ij}^{\alpha\beta}\right) = 0 \quad \text{on } \partial\Omega_{ij}^{\downarrow i} \text{ and } \partial\Omega_{\alpha\beta}^{\uparrow\beta} \tag{7.12}$$

with a parameter vector $\mathbf{p}_{ij}^{(\alpha\beta)} \in \Re^{k_{ij}^{\alpha\beta}}$.

The transport laws for the C^0-discontinuous flow $\mathbf{\Phi}(\mathbf{x}, t)$ on the boundary $\partial\Omega_{ij}$ are sketched in Fig. 7.8 for explaining the above definition. In the domain Ω_α ($\alpha \in \{i, j\}$), there is a flow $\mathbf{\Phi}^{(\alpha)}(\mathbf{x}, t)$ from the starting point $\mathbf{x}_\kappa^{(\alpha)}$ to the ending point $\mathbf{x}_{\kappa+1}^{(\beta)}$ where $k \in \{m-1, m, m+1\}$. Note that the superscripts "−" and "+" represent before and after the transport on the boundary. The dashed curves represent the transport law on the boundary. The solid thin curves represent the subdomains. In Fig. 7.8(a), the transport laws are given by the following two equations,

$$T_{ij} : \mathbf{x}_{m-}^{(i)} \to \mathbf{x}_{m+}^{(i)} \qquad \text{on } \overrightarrow{\partial\Omega}_{ij}; \tag{7.13}$$

$$T_{ji} : \mathbf{x}_{(m+1)-}^{(j)} \to \mathbf{x}_{(m+1)+}^{(i)} \qquad \text{on } \overleftarrow{\partial\Omega}_{ij}. \tag{7.14}$$

The two transport laws transport the flow on the same boundary with different domains. For the transport law T_{ij}, $\partial\Omega_{ij}^{\downarrow i}, \partial\Omega_{ij}^{\uparrow j} \subset \overrightarrow{\partial\Omega}_{ij}$ are for the C^0-discontinuous system on the semi-passable boundary $\overrightarrow{\partial\Omega}_{ij}$. However, for the

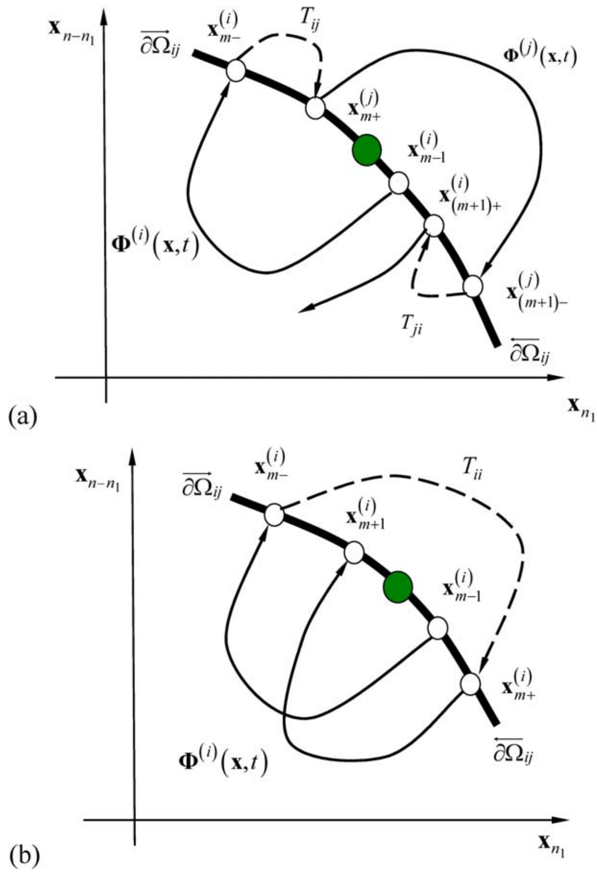


Figure 7.8. Transport laws for the C^0-discontinuous flow $\mathbf{\Phi}(\mathbf{x}, t)$ on the boundary $\partial\Omega_{ij}$: (a) from Ω_α to Ω_β on $\partial\Omega_{\alpha\beta}^{\downarrow\alpha}$ ($\alpha, \beta \in \{i, j\}$ and $\alpha \neq \beta$), and (b) from $\partial\Omega_{ij}^{\downarrow i}$ to $\partial\Omega_{ji}^{\uparrow i}$.

transport law T_{ji}, $\partial\Omega_{ij}^{\downarrow j}, \partial\Omega_{ij}^{\uparrow i} \subset \overleftarrow{\partial\Omega}_{ij}$. To make the expression different from nonpassable boundary, the input and output boundary of $\overrightarrow{\partial\Omega}_{ij}$ can be expressed by $\overrightarrow{\partial\Omega}_{ij}^{\downarrow i}$ and $\overrightarrow{\partial\Omega}_{ij}^{\uparrow j}$, respectively. The input and output boundaries of $\overline{\partial\Omega}_{ij}$ can also be expressed accordingly, by $\overline{\partial\Omega}_{ij}^{\downarrow i}$ and $\overline{\partial\Omega}_{ij}^{\uparrow j}$. Without the transport law, the flow passability is the same as in Chapters 2–6. In Fig. 7.8(b), the transport law maps a point $\mathbf{x}_{m-}^{(i)} \in \partial\Omega_{ij}^{\downarrow i} \subset \overrightarrow{\partial\Omega}_{ij}$ to the point $\mathbf{x}_{m+}^{(i)} \in \partial\Omega_{ij}^{\uparrow i} \subset \overleftarrow{\partial\Omega}_{ij}$ at time t_m without time consumption, i.e.,

$$T_{ii} : \mathbf{x}_{m-}^{(i)} \to \mathbf{x}_{m+}^{(i)} \quad \text{on the side of } \Omega_i \text{ for } \overleftrightarrow{\partial\Omega}_{ij}. \tag{7.15}$$

After transported, the flow is still on the side of Ω_i. That is, the motion cannot pass over the boundary $\partial\Omega_{ij}$ into the domain Ω_j since the transport law causes the motion into the returning semi-passable boundary to avoid the flow in the domain Ω_j. The discontinuous transport law causes the passable boundary equivalent to the nonpassable boundary. Once Definition 7.12 is extended, the transport law maps the one accessible domain boundary to another nonadjacent domain boundary. A similar definition can be given as follows:

DEFINITION 7.13. For a discontinuous system in Eq. (2.1), under Assumptions (A1)–(A3), there are two nonadjacent accessible domains Ω_i and Ω_l with the corresponding i-side input and l-side output boundaries $\partial\Omega_{ij}^{\downarrow i}$ and $\partial\Omega_{kl}^{\uparrow l}$. A map T_{il} transporting $\mathbf{x}_{m-}^{(i)} \in \partial\Omega_{ij}^{\downarrow i}$ to $\mathbf{x}_{m+}^{(l)} \in \partial\Omega_{kl}^{\uparrow l}$ with $\mathbf{x}_{m-}^{(i)} \neq \mathbf{x}_{m+}^{(l)}$ for time t_m without the time consumption is termed a transport law from $\partial\Omega_{ij}^{\downarrow i}$ to $\partial\Omega_{kl}^{\uparrow l}$, i.e.,

$$T_{il} : \mathbf{x}_{m-}^{(i)} \to \mathbf{x}_{m+}^{(l)} \tag{7.16}$$

which is governed by the $(n-1)$-dimensional one-to-one vector function $\mathbf{g}_{il} \in \Re^{n-1}$ as

$$g_{il}\big(\mathbf{x}_{m-}^{(i)}, \mathbf{x}_{m+}^{(l)}, \mathbf{p}_{ij}^{kl}\big) = 0 \quad \text{on } \partial\Omega_{ij}^{\downarrow i} \text{ and } \partial\Omega_{kl}^{\uparrow k} \tag{7.17}$$

with a parameter vector $\mathbf{p}_{ij}^{(kl)} \in \Re^{k_{ij}^{kl}}$.

The Definition 7.12 is the special case of Definition 7.13 when $k, l \in \{i, j\}$ and $k \neq l$. From this definition, it is not necessary to require the domains Ω_j and Ω_k to be accessible. They can be accessible, or inaccessible or universal domain. This definition is illustrated in Fig. 7.9. As in Section 7.1, the boundary may be permanently nonpassable. To continue the flow in the entire discontinuous system, the transport law should be exerted. In the above transport laws, the mappings are time-independent. For time-dependent transport laws, they are as similar as regular dynamical systems. The difference lies in that the flow of the transport law is given instead of being determined by the differential equations. Therefore, the time-dependent transport law is introduced as:

DEFINITION 7.14. For a discontinuous system in Eq. (2.1), under Assumptions (A1)–(A3), there are two nonadjacent accessible domains Ω_i and Ω_l with the corresponding i-side input and l-side output boundaries $\partial\Omega_{ij}^{\downarrow i}$ and $\partial\Omega_{kl}^{\uparrow l}$. A map ${}^tT_{il}$ transporting $\mathbf{x}_{m-}^{(i)} \in \partial\Omega_{ij}^{\downarrow i}$ with time t_m to $\mathbf{x}_{(m+1)+}^{(l)} \in \partial\Omega_{kl}^{\uparrow l}$ with t_{m+1} is termed a transport law from $\partial\Omega_{ij}^{\downarrow i}$ to $\partial\Omega_{kl}^{\uparrow l}$ with a given time difference $\Delta t = t_{m+1} - t_m$, i.e.,

$${}^tT_{il} : \big(\mathbf{x}_{m-}^{(i)}, t_m\big) \to \big(\mathbf{x}_{(m+1)+}^{(l)}, t_{m+1}\big) \tag{7.18}$$

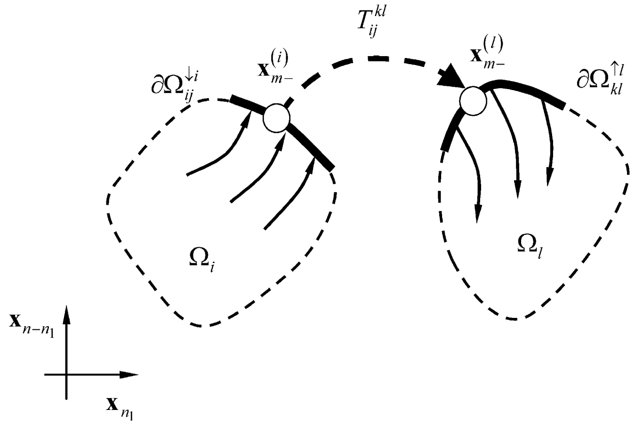


Figure 7.9. A generalized transport laws from $\partial\Omega_{ij}^{\downarrow i}$ to $\partial\Omega_{kl}^{\uparrow l}$ for the C^0-discontinuous flow.

which is governed by a vector function with two components

$$^{t}\mathbf{g}_{il}\big(\mathbf{x}_{m-}^{(i)}, \mathbf{x}_{m+}^{(l)}, \mathbf{p}_{ij}^{kl}, t_m, t_{m+1}\big) = 0 \quad \text{on } \partial\Omega_{ij}^{\downarrow i} \text{ and } \partial\Omega_{kl}^{\uparrow k} \tag{7.19}$$

with a parameter vector $\mathbf{p}_{ij}^{(kl)} \in \Re^{k_{ij}^{kl}}$.

7.3. Mapping dynamics

For a discontinuous system, if a flow just in the single subdomain cannot be intersected with any discontinuous boundary, such a case will not be discussed in this book. This is because the characteristics of such a flow can be determined by the continuous dynamical system theory. The main focus of this chapter is to determine the dynamical properties of global flows intersected with discontinuous boundaries in discontinuous dynamical systems. To do so, the naming of subdomains and boundaries in discontinuous dynamic systems is very crucial. Once the domains and boundaries in discontinuous dynamical systems are named, the mappings and mapping structures for the global flow in such discontinuous dynamical systems can be developed. Therefore, consider a universal domain $\mho$ with M-subdomains in phase space, and N-boundaries among the M-subdomains with the universal domains. In the previous discussion, the boundaries are expressed by the neighbored domains. For example, $\partial\Omega_{IJ}$ is the boundary between the subdomains Ω_I and Ω_J. To define the maps, the switching surfaces relative to the boundary should be named, and the domain should be named. The naming of

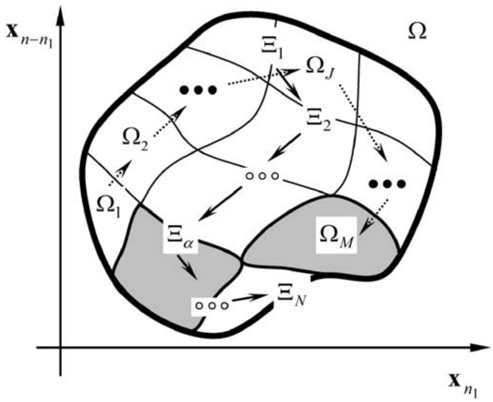


Figure 7.10. Naming subdomains and boundaries: the dotted routine for the order of naming for subdomains, and the solid routine for switching surfaces.

switching surfaces and subdomains can be independent. In Fig. 7.10, the subdomains are named through a dotted routine, and the switching planes are named by a solid routine. In fact, the naming of the subdomains and the switching planes are arbitrary. Consider all the subdomains and the switching surfaces expressed by Ω_J $(J = 1, 2, \ldots, M)$ and Ξ_α $(\alpha = 1, 2, \ldots, N)$, respectively. The switching plane Ξ_α $(\alpha = 1, 2, \ldots, N)$ defined on the boundary $\partial\Omega_{IJ}$ $(I, J = 1, 2, \ldots, M)$ are given by

$$\Xi_\alpha = \left\{(t_i, \mathbf{x}_i) \mid \varphi_{IJ}(\mathbf{x}_i, t_i) = 0 \text{ for time } t_i\right\} \subset \partial\Omega_{IJ}. \tag{7.20}$$

The local mapping relative to the switching sets Ξ_α through the dynamical system in the subdomain Ω_J is

$$P_{J_{\alpha\alpha}} : \Xi_\alpha \to \Xi_\alpha. \tag{7.21}$$

The global mapping starting on the switching sets Ξ_α and ending on one of the rest switching sets Ξ_β $(\beta \neq \alpha)$ relative to the subdomain Ω_J is

$$P_{J_{\alpha\beta}} : \Xi_\alpha \to \Xi_\beta. \tag{7.22}$$

The sliding mapping on the switching sets Ξ_α governed by the boundary $\varphi_{IJ}(\mathbf{x}, t) = 0$ is defined by

$$P_{0_{\alpha\alpha}} : \Xi_\alpha \to \Xi_\alpha. \tag{7.23}$$

Notice that no mapping can be defined for the inaccessible domain. For simplicity, the following notation for mapping is introduced:

$$P_{n_k \cdots n_i \cdots n_2 n_1} = P_{n_k} \circ \cdots \circ P_{n_i} \circ \cdots \circ P_{n_2} \circ P_{n_1} \tag{7.24}$$

where all the local and global mappings are

$$P_{n_i} \in \left\{ P_{J_{\alpha\beta}} \mid \alpha, \beta \in \{1, 2, \ldots, N\}, J \in \{0, 1, 2, \ldots, M\} \right\} \tag{7.25}$$

for $i \in \{1, 2, \ldots, k\}$. The rotation of the mapping of periodic motion in order gives the same motion (i.e., $P_{n_1 n_2 \cdots n_k}, P_{n_2 \cdots n_k n_1}, \ldots, P_{n_k n_1 \cdots n_{k-1}}$), and only the selected Poincaré mapping section is different. The motion of the *m-time* repeating of mapping $P_{n_1 n_2 \cdots n_k}$ is defined as

$$\begin{aligned} P^m_{n_k \cdots n_2 n_1} &\equiv P_{(n_k \cdots n_2 n_1)^m} \equiv P_{\underbrace{(n_k \cdots n_2 n_1) \cdots (n_k \cdots n_2 n_1)}_{m}} \\ &\equiv \underbrace{(P_{n_k} \circ \cdots \circ P_{n_2} \circ P_{n_1}) \circ \cdots \circ (P_{n_k} \circ \cdots \circ P_{n_2} \circ P_{n_1})}_{m \text{ sets}}. \end{aligned} \tag{7.26}$$

To extend this concept to the local mapping, define

$$P^m_{n_l \cdots n_j} \equiv P_{(n_l \cdots n_j)^m} \equiv \underbrace{(P_{n_l} \circ \cdots \circ P_{n_j}) \circ \cdots \circ (P_{n_l} \circ \cdots \circ P_{n_j})}_{m \text{ sets}} \tag{7.27}$$

where the local mappings are

$$P_{n_{j_1}} \in \left\{ P_{J_{\alpha\alpha}} \mid \alpha \in \{1, 2, \ldots, N\}, J \in \{0, 1, 2, \ldots, M\} \right\} \tag{7.28}$$

for $j_1 \in \{j, \ldots, l\}$. Notice that the Jth subdomain for the local mappings should be the neighbored domains of the switching set Ξ_α. For a special combination of global and local mapping, introduce a mapping structure

$$\begin{aligned} P_{n_k \cdots (n_l \cdots n_j)^m \cdots n_2 n_1} &\equiv P_{n_k} \circ \cdots \circ P^m_{n_l \cdots n_j} \circ \cdots \circ P_{n_2} \circ P_{n_1} \\ &= P_{n_k} \circ \cdots \circ \underbrace{(P_{n_l} \circ \cdots \circ P_{n_j}) \circ \cdots \circ (P_{n_l} \circ \cdots \circ P_{n_j})}_{m \text{ sets}} \circ \cdots \circ P_{n_2} \circ P_{n_1}. \end{aligned} \tag{7.29}$$

From the definition, any global flow of the dynamical systems in Eq. (2.1) can be very easily managed through a certain mapping structures accordingly. Further, all the periodic flow of such a system in Eq. (2.1) can be investigated.

For a subdomain Ω_J, the flow is given by

$$\mathbf{x}^{(J)} = \boldsymbol{\Phi}^{(J)}\big(t, \mathbf{x}_0^{(J)}, t_0\big) \in \Re^n. \tag{7.30}$$

Consider an initial condition to be chosen on the discontinuous boundary relative to the switching plane Ξ_α (i.e., $(\mathbf{x}_i, t_i)$). Once a flow in the subdomain Ω_J for time $t_{i+1} > t_i$ arrives to the boundary with the switching plane Ξ_β (i.e., $(\mathbf{x}_{i+1}, t_{i+1})$), Eq. (7.30) becomes

$$\mathbf{x}_{i+1}^{(J)} = \boldsymbol{\Phi}^{(J)}\big(t_{i+1}, \mathbf{x}_i^{(J)}, t_i\big) \in \Re^n. \tag{7.31}$$

The foregoing vector equation gives the relationships for mapping $P_{J_{\alpha\beta}}$, which maps the starting point $(\mathbf{x}_i, t_i)$ to the final point $(\mathbf{x}_{i+1}, t_{i+1})$ in the subdomain

Ω_J. For an n-dimensional dynamical system, Eq. (7.31) gives n scalar equations. For one-degree-of-freedom systems, two scalar algebraic equations will be given. For a sliding flow on the switching plane Ξ_α, the sliding mapping $P_{0_{\alpha\alpha}}$ with starting and ending points $(\mathbf{x}_i, t_i)$ and $(\mathbf{x}_{i+1}, t_{i+1})$. Consider an $(n-1)$-dimensional boundary $\partial\Omega_{IJ}$, on which the switching set Ξ_α is defined. For the sliding mapping $P_{0_{\alpha\alpha}} : \Xi_\alpha \to \Xi_\alpha$, the starting and ending points satisfy $\varphi_{IJ}(\mathbf{x}_i, t_i) = \varphi_{IJ}(\mathbf{x}_{i+1}, t_{i+1}) = 0$. The sliding dynamics on the boundary can be determined by Eq. (3.6), i.e.,

$$\dot{\mathbf{x}} = \mathbf{F}_{IJ}^{(0_{\alpha\alpha})}(\mathbf{x}, t) \in \Re^{n-1} \quad \text{on } \partial\Omega_{IJ}. \tag{7.32}$$

With the starting point $(\mathbf{x}_i, t_i)$, Eq. (7.32) gives

$$\mathbf{x} = \mathbf{G}^{(0_{\alpha\alpha})}(t, \mathbf{x}_i, t_i) \quad \text{and} \quad \varphi_{IJ}(\mathbf{x}, t) = 0. \tag{7.33}$$

From the vanishing conditions of the sliding motion on the boundary, at the ending point, Eq. (7.33) should be satisfied and the normal vector field product should be zero. Therefore the sliding mapping $P_{0_{\alpha\alpha}}$ on the boundary with $\varphi_{IJ}(\mathbf{x}_{i+1}, t_{i+1}) = 0$ will be governed by

$$\begin{aligned} &\mathbf{x}_{i+1} = \mathbf{G}^{(0_{\alpha\alpha})}(t_{i+1}, \mathbf{x}_i, t_i) \in \Re^{n-1}, \\ &\mathbf{n}_{\partial\Omega_{IJ}}^{\mathrm{T}} \cdot \mathbf{F}^{(\sigma)}(\mathbf{x}_{i+1}, t_{i+1}) = 0, \quad \sigma \in \{I, J\}. \end{aligned} \tag{7.34}$$

Equation (7.34) can be rewritten in a general form of

$$\mathbf{x}_{i+1}^{(0_{\alpha\alpha})} = \mathbf{\Phi}^{(0_{\alpha\alpha})}\big(t_{i+1}, \mathbf{x}_i^{(0_{\alpha\alpha})}, t_i\big) \in \Re^n. \tag{7.35}$$

The transport law can be defined as a transport mapping as $P_{T_{\alpha\beta}} : \Xi_\alpha \to \Xi_\beta$. Similarly, using $t_{i+1} - t_i = \Delta$ the transport law gives

$$\mathbf{x}_{i+1}^{(T_{\alpha\alpha})} = \mathbf{\Phi}^{(T_{\alpha\alpha})}\big(t_{i+1}, \mathbf{x}_i^{(T_{\alpha\alpha})}, t_i\big) \in \Re^n. \tag{7.36}$$

For practical computations, the transport law is not necessary to be treated as mapping. Once all the single mappings are determined by the corresponding governing equation. Because of the global flow on the discontinuous boundary in phase space, a new vector $\mathbf{y} = (t, \mathbf{x})^{\mathrm{T}}$ is selected on the boundary. Through the new vectors and boundaries, the global flow based on the mapping structure is expressed by

$$\mathbf{y}_{i+r} = P_{n_k \cdots (n_l \cdots n_j)^m \cdots n_2 n_1} \mathbf{y}_i \tag{7.37}$$

where r is the total number of mapping actions in the mapping structure. For a global periodic flow, the periodicity conditions are required by

$$(t_{i+r}, \mathbf{x}_{i+r}) = (t_i + NT, \mathbf{x}_i) \tag{7.38}$$

where N is positive integer, and T is period of system in Eq. (2.1). For system without external periodic vector fields, the flow with a certain time difference returns to the selected reference plane, which will be a periodic motion. The governing equations for Eq. (7.37) are

$$\begin{aligned} &\mathbf{x}_{i+\rho}^{(\sigma)} = \boldsymbol{\Phi}^{(J)}\big(t_{i+\rho}, \mathbf{x}_{i+\rho-1}^{(\sigma)}, t_{i+\rho-1}\big), \\ &\varphi_{IJ}\big(\mathbf{x}_{i+\rho-1}^{(\sigma)}, t_{i+\rho-1}\big) = 0 \end{aligned} \tag{7.39}$$

for $\sigma \in \{n_1, n_2, \ldots, n_k\}$, $\rho \in \{1, 2, \ldots, r\}$ and $I, J = 1, 2, \ldots, M$.

The global periodic flow relative to the mapping structure $P_{n_k \cdots (n_l \cdots n_j)^m \cdots n_2 n_1}$ will be determined by Eqs. (7.38) and (7.39). The global periodic flow may be stable and unstable. The corresponding stability analysis can be completed through the traditional local stability analysis. For the periodic flow with sliding motion, the local stability analysis may not be useful. The sliding criteria in Chapter 3 should be employed. Although the local stability analysis can be carried out, it cannot provide enough information to judge the disappearance of the certain global, periodic flow. For the local stability analysis of the periodic flow, the all switching points are given by $(\mathbf{x}_{i+\rho-1}^{(\sigma)}, t_{i+\rho-1}^{(\sigma)})$ and the corresponding perturbations $\delta\mathbf{y}_{i+\rho-1}^{(\sigma)} = (\delta\mathbf{x}_{i+\rho-1}^{(\sigma)}, \delta t_{i+\rho-1}^{(\sigma)})^{\mathrm{T}}$ are adopted. The perturbed equation for the stable analysis is

$$\delta\mathbf{y}_{i+r} = \mathrm{D}P_{n_k \cdots (n_l \cdots n_j)^m \cdots n_2 n_1} \delta\mathbf{y}_i, \tag{7.40}$$

and the Jacobian matrix is

$$\mathrm{D}P_{n_k \cdots (n_l \cdots n_j)^m \cdots n_2 n_1} = \mathrm{D}P_{n_k} \cdot \cdots \cdot (\mathrm{D}P_{n_l} \cdot \cdots \cdot \mathrm{D}P_{n_j})^m \cdot \cdots \cdot \mathrm{D}P_{n_2} \cdot \mathrm{D}P_{n_1}. \tag{7.41}$$

For each single mapping,

$$\mathrm{D}P_{\sigma} = \left[\frac{\partial(\mathbf{x}_{i+\rho}^{(\sigma)}, t_{i+\rho})}{\partial(\mathbf{x}_{i+\rho-1}^{(\sigma)}, t_{i+\rho-1})} \right]_{(\mathbf{x}_{i+\rho}^{(\sigma)}, t_{i+\rho})} \tag{7.42}$$

for $\sigma \in \{n_1, n_2, \ldots, n_k\}$, $\rho \in \{1, 2, \ldots, r\}$ and $I, J = 1, 2, \ldots, M$. The following determinant gives the all eigenvalues to determine the stability, i.e.,

$$|\mathrm{D}P_{n_k \cdots (n_l \cdots n_j)^m \cdots n_2 n_1} - \lambda\mathbf{I}| = 0. \tag{7.43}$$

If the magnitude of each eigenvalue λ_j $(j = 1, \ldots, s)$ is less than one (i.e., $|\lambda_j| < 1$), the periodic flow determined by Eqs. (7.38) and (7.39) is stable. If at least one of the magnitudes of s eigenvalues is greater than one (i.e., $|\lambda_j| > 1$, $j \in \{1, \ldots, s\}$), the periodic flow is unstable. If all the eigenvalues are on the unit circle, the periodic flow is spectrally stable.

The mapping dynamics is to develop the mapping structure for aspecific motions in nonlinear dynamical system. The mapping structure technique is the

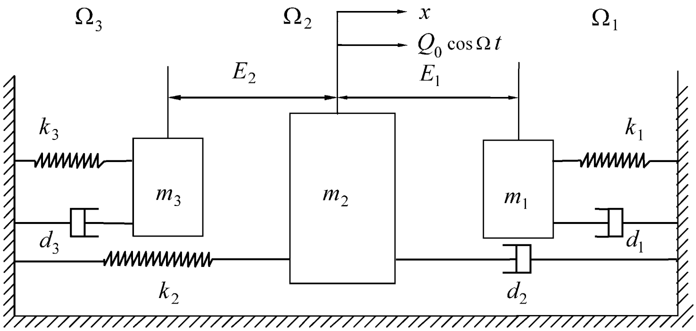


Figure 7.11. Mechanical model of the piecewise linear impacting oscillator.

extension of the shooting method. For complicated motions, the shooting method may have difficulties. The details on the shooting approach can be found in Levinson (1949).

7.4. An impacting piecewise system

Consider a periodically forced piecewise linear oscillator with impacting at two displacement boundaries in Fig. 7.11. In this system, three masses m_J ($J = 1, 2, 3$) are connected with the corresponding dampers with coefficients d_J and springs of stiffness k_J. The external forcing $Q_0 \cos \Omega t$ acts on the primary mass m_2 where Q_0 and Ω are excitation amplitude and frequency, respectively. The displacement measured from the equilibrium position is denoted by x. To develop this model for transmission gear systems, the following hypotheses are used:

(H1) The two second masses m_J ($J = 1, 3$) are stationary before impact with the primary mass m_2.
(H2) The impact between the primary mass m_2 and the second mass m_J ($J = 1, 3$) is perfectly plastic at two displacement boundaries. During impacting at the separation boundaries, time is invariant.
(H3) Once the primary and second masses return back to the displacement boundary, the second mass m_J ($J = 1, 3$) is separated with the primary mass m_2.

After and before impacts, the motion of the system has been changed. Therefore, three equations of motion will be used to describe the entire dynamical system. The equations of motion for this system are:

$$\ddot{x} + 2d_J\dot{x} + c_Jx = A_J \cos \Omega t - b_J, \quad J \in \{1, 2, 3\} \tag{7.44}$$

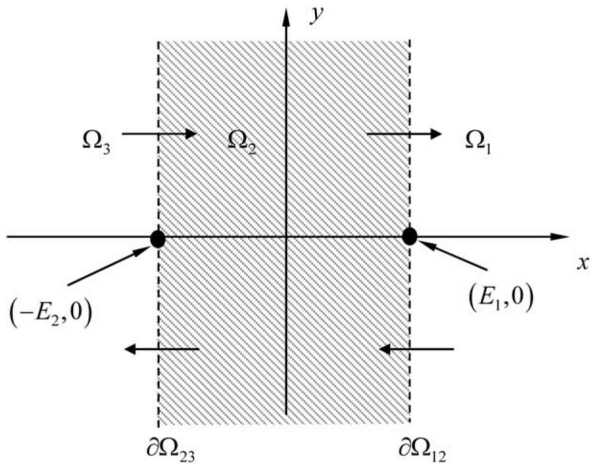


Figure 7.12. Phase domain partition and boundaries.

where

$$A_J = \frac{Q_0}{m_2 + m_J(1-\delta_J^2)}, \quad c_J = \frac{k_2 + k_J(1-\delta_J^2)}{m_2 + m_J(1-\delta_J^2)},$$
$$d_J = \frac{d_2 + d_J(1-\delta_J^2)}{m_2 + m_J(1-\delta_J^2)}, \quad b_J = \frac{(2-J)k_J E_J}{m_2 + m_J(1-\delta_J^2)}. \tag{7.45}$$

Note that the Kronecker function $\delta_I^J = 0$ for $I \neq J$ and $\delta_I^J = 1$ for $I = J$. The three subdomains in phase space for the discontinuous system in Eq. (7.44) are defined as

$$\Omega = \bigcup_{J=1}^{3} \Omega_J \tag{7.46}$$

where

$$\begin{aligned} \Omega_1 &= \{(x,y) \mid x \in [E_1,\infty), y \in (-\infty,\infty)\}, \\ \Omega_2 &= \{(x,y) \mid x \in [-E_2, E_1], y \in (-\infty,\infty)\}, \\ \Omega_3 &= \{(x,y) \mid x \in (-\infty,-E_2], y \in (-\infty,\infty)\}. \end{aligned} \tag{7.47}$$

The corresponding separation boundaries are defined as

$$\begin{aligned} \partial\Omega_{12} &= \overrightarrow{\partial\Omega}_{12} \cup \overleftarrow{\partial\Omega}_{12} \cup (E_1,0) = \Omega_1 \cap \Omega_2 \\ &= \{(x,y) \mid \varphi_{12}(x,y) \equiv x - E_1 = 0\}, \\ \partial\Omega_{23} &= \overrightarrow{\partial\Omega}_{23} \cup \overleftarrow{\partial\Omega}_{23} \cup (-E_2,0) = \Omega_2 \cap \Omega_3 \end{aligned} \tag{7.48}$$

$$= \left\{(x, y) \mid \varphi_{23}(x, y) \equiv x + E_2 = 0\right\}.$$

For a better illustration of the above definition, the subdomains and boundaries are shown in Fig. 7.12. From the above definitions, Eqs. (7.44) and (7.45) give

$$\dot{\mathbf{x}} = \mathbf{F}^{(J)}(\mathbf{x}, t) \quad \text{for } J \in \{1, 2, 3\} \tag{7.49}$$

where $\mathbf{x} = (x, y)^{\mathrm{T}}$ and

$$\mathbf{F}^{(j)}(\mathbf{x}, t) = (y, -2d_J y - c_J x - b_J + Q_0 \cos \Omega t)^{\mathrm{T}} \quad \text{in } \mathbf{x} \in \Omega_J. \tag{7.50}$$

From three assumptions and momentum conservation, the transport law for this discontinuous dynamical system is given by

$$y(t_{m+}) = \mu_{IJ}\, y(t_{m-}) \quad \text{on } \partial\Omega_{IJ} \tag{7.51}$$

where t_{m-} and t_{m+} are times of before and after impacts, and

$$\mu_{12} = \frac{m_2}{m_1 + m_2}, \qquad \mu_{23} = \frac{m_2}{m_2 + m_3}. \tag{7.52}$$

Using Eq. (7.48), the normal vector of the discontinuous boundary is

$$\mathbf{n}_{\partial\Omega_{12}} = \mathbf{n}_{\partial\Omega_{23}} = (1, 0)^{\mathrm{T}}. \tag{7.53}$$

The normal vectors of the boundary indicate the separation boundary convex to the direction of the x-direction in phase space. That is, the boundaries $\partial\Omega_{12}$ and $\partial\Omega_{23}$ are convex to the domain Ω_1 and Ω_2, respectively. Therefore, Eqs. (7.50) and (7.53) give

$$\begin{aligned} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot \mathbf{F}^{(1)}(\mathbf{x}, t, \boldsymbol{\mu}_1, \boldsymbol{\pi}) &= \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot \mathbf{F}^{(2)}(\mathbf{x}, t, \boldsymbol{\mu}_2, \boldsymbol{\pi}) = y, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{23}} \cdot \mathbf{F}^{(2)}(\mathbf{x}, t, \boldsymbol{\mu}_2, \boldsymbol{\pi}) &= \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{23}} \cdot \mathbf{F}^{(3)}(\mathbf{x}, t, \boldsymbol{\mu}_3, \boldsymbol{\pi}) = y. \end{aligned} \tag{7.54}$$

The total derivative of vector fields is determined by

$$\mathrm{D}\mathbf{F}^{(J)}(t_{m\pm}) = \left[\frac{\partial F_p^{(J)}(t_{m\pm})}{\partial x_q}\right]\mathbf{F}^{(\alpha)}(t_{m\pm}) + \frac{\partial \mathbf{F}^{(J)}(t_{m\pm})}{\partial t}. \tag{7.55}$$

Therefore,

$$\begin{aligned} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot \mathrm{D}\mathbf{F}^{(1)}(\mathbf{x}, t, \boldsymbol{\mu}_1, \boldsymbol{\pi}) &= F^{(1)}(\mathbf{x}, t, \boldsymbol{\mu}_1, \boldsymbol{\pi}), \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot \mathrm{D}\mathbf{F}^{(2)}(\mathbf{x}, t, \boldsymbol{\mu}_2, \boldsymbol{\pi}) &= F^{(2)}(\mathbf{x}, t, \boldsymbol{\mu}_2, \boldsymbol{\pi}); \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{23}} \cdot \mathrm{D}\mathbf{F}^{(2)}(\mathbf{x}, t, \boldsymbol{\mu}_2, \boldsymbol{\pi}) &= F^{(2)}(\mathbf{x}, t, \boldsymbol{\mu}_2, \boldsymbol{\pi}), \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{23}} \cdot \mathrm{D}\mathbf{F}^{(3)}(\mathbf{x}, t, \boldsymbol{\mu}_3, \boldsymbol{\pi}) &= F^{(3)}(\mathbf{x}, t, \boldsymbol{\mu}_3, \boldsymbol{\pi}). \end{aligned} \tag{7.56}$$

From Theorem 2.11, Eqs. (2.29) and (2.30) give the necessary and sufficient conditions for the tangential (or grazing) bifurcation on the separation boundary, i.e.,

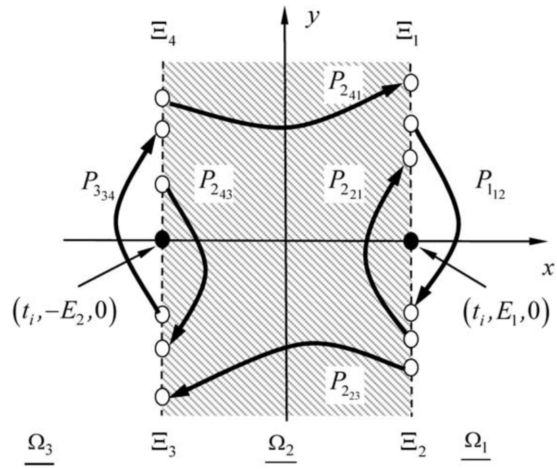


Figure 7.13. Switching sections and basic mappings in phase plane.

$$y(t_m) = 0, \quad \text{on } \partial\Omega_{12} \text{ and } \partial\Omega_{23}; \tag{7.57}$$

$$\begin{aligned} &F^{(1)}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_1, \boldsymbol{\pi}) > 0, \quad F^{(2)}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_2, \boldsymbol{\pi}) < 0 \quad \text{on } \partial\Omega_{12}, \\ &F^{(2)}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_2, \boldsymbol{\pi}) > 0, \quad F^{(3)}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_3, \boldsymbol{\pi}) < 0 \quad \text{on } \partial\Omega_{23}. \end{aligned} \tag{7.58}$$

To describe motion in Eq. (7.44), the generic mapping should be defined. Four switching sections (or sets) are introduced first:

$$\begin{aligned} \Xi_1 &= \left\{(t_i, x_i, y_i) \mid x_i - E_1 = 0, \dot{x}_i = y_i > 0\right\} \subset \overleftarrow{\partial\Omega}_{12}, \\ \Xi_2 &= \left\{(t_i, x_i, y_i) \mid x_i - E_1 = 0, \dot{x}_i = y_i < 0\right\} \subset \overrightarrow{\partial\Omega}_{12}, \\ \Xi_3 &= \left\{(t_i, x_i, y_i) \mid x_i + E_2 = 0, \dot{x}_i = y_i < 0\right\} \subset \overrightarrow{\partial\Omega}_{23}, \\ \Xi_4 &= \left\{(t_i, x_i, y_i) \mid x_i + E_2 = 0, \dot{x}_i = y_i > 0\right\} \subset \overleftarrow{\partial\Omega}_{23}. \end{aligned} \tag{7.59}$$

The two unions of the switching planes are:

$$\Xi_{12} = \Xi_1 \cup \Xi_2 \cup (t_i, E_1, 0) \quad \text{and} \quad \Xi_{23} = \Xi_3 \cup \Xi_4 \cup (t_i, -E_2, 0). \tag{7.60}$$

The points $(t_i, E_1, 0)$ and $(t_i, -E_2, 0)$ are both equilibriums of the sliding motion and the tangential points for the flows on the both sides of the separation boundary. From four subsets, six generic mappings are:

$$P_{J_{\alpha\beta}} : \Xi_\alpha \to \Xi_\beta \quad \text{for } J_{\alpha\beta} = 1_{12}, 2_{21}, 2_{23}, 3_{34}, 2_{43}, 2_{41}. \tag{7.61}$$

The switching planes and generic mappings are sketched in Fig. 7.13. Consider the initial and final states $(t, x, \dot{x}) \equiv (t_i, x_i, y_i)$ and $(t_{i+1}, x_{i+1}, y_{i+1})$ in the three subdomains Ω_J $(J = 1, 2, 3)$, respectively. The displacement and velocity equations in Appendix A with the initial conditions give the governing equations for mapping $P_{J_{\alpha\beta}}$ $(J_{\alpha\beta} = 1_{12}, 2_{21}, 2_{23}, 3_{34}, 2_{43}, 2_{41})$, i.e.,

$$P_{J_{\alpha\beta}}\colon \begin{cases} f_1^{(J_{\alpha\beta})}(x_i, y_i, t_i, x_{i+1}, y_{i+1}, t_{i+1}) = 0 \\ f_2^{(J_{\alpha\beta})}(x_i, y_i, t_i, x_{i+1}, y_{i+1}, t_{i+1}) = 0 \end{cases} \tag{7.62}$$

where $x_i, x_{i+1} \in \{E_1, -E_2\}$.

Due to impacts at two boundaries, the transport law relations are

$$\begin{aligned} y_i^+ &= \mu_{21} y_i^-, \quad \text{for } x_i = x_{i+1} = E_1, \\ y_i^+ &= \mu_{23} y_i^-, \quad \text{for } x_i = x_{i+1} = -E_2 \end{aligned} \tag{7.63}$$

where $y_i^- \equiv y_{i+1}$ for mappings $P_{2_{23}}$ and $P_{2_{41}}$ before impact, and $y_i^+ \equiv y_i$ for mappings $P_{1_{12}}$ and $P_{3_{34}}$ after impact. In addition, from Eqs. 7.57 and 7.58, the necessary and sufficient conditions for all the six generic mappings $P_{J_{\alpha\beta}}$ $(J_{\alpha\beta} = 1_{12}, 2_{21}, 2_{23}, 3_{34}, 2_{43}, 2_{41})$ in the three domains Ω_J $(J \in \{1, 2, 3\})$ are

$$\begin{aligned} & y_{i+1} = 0; \\ & \dot{y}_{i+1} = -c_J x_{i+1} - b_J + Q_0 \cos \Omega t_{i+1} > 0 \\ & \qquad \text{for } P_{J_{\alpha\beta}} \ (J_{\alpha\beta} = 1_{12}, 2_{23}, 2_{41}), \\ & \dot{y}_{i+1} = -c_J x_{i+1} - b_J + Q_0 \cos \Omega t_{i+1} < 0 \\ & \qquad \text{for } P_{J_{\alpha\beta}} \ (J_{\alpha\beta} = 2_{21}, 3_{34}, 2_{41}). \end{aligned} \tag{7.64}$$

Once the generic mappings are introduced, the mapping structure of periodic motion in this discontinuous system is discussed. Consider a periodic motion with the mapping structure described in Fig. 7.14, i.e.,

$$P_{2_{41}(3_{34}2_{43})^{k_2}3_{34}2_{23}(1_{12}2_{21})^{k_1}1_{12}} \equiv P_{2_{41}} \circ P^{(k_2)}_{3_{34}2_{43}} \circ P_{3_{34}} \circ P_{2_{23}} \circ P^{(k_1)}_{1_{12}2_{21}} \circ P_{1_{12}}. \tag{7.65}$$

For $k_2 = k_1 = 0$, the foregoing mapping structure becomes the simplest one as

$$P_{2_{41}3_{34}2_{23}1_{12}} \equiv P_{2_{41}} \circ P_{3_{34}} \circ P_{2_{23}} \circ P_{1_{12}}. \tag{7.66}$$

On the other hand, a generalized mapping structure is

$$\begin{aligned} & P_{2_{41}(3_{34}2_{43})^{k_{2n}}3_{34}2_{23}(1_{12}2_{21})^{k_{1n}}1_{12}\cdots 2_{41}(3_{34}2_{43})^{k_{2m}}3_{34}2_{23}(1_{12}2_{21})^{k_{1m}}1_{12}\cdots 2_{41}(3_{34}2_{43})^{k_{21}}3_{34}2_{23}(1_{12}2_{21})^{k_{11}}1_{12}} \\ & \quad \equiv P_{2_{41}(3_{34}2_{43})^{k_{2n}}3_{34}2_{23}(1_{12}2_{21})^{k_{1n}}1_{12}} \circ \cdots \circ P_{2_{41}(3_{34}2_{43})^{k_{2m}}3_{34}2_{23}(1_{12}2_{21})^{k_{1m}}1_{12}} \\ & \qquad \circ \cdots \circ P_{2_{41}(3_{34}2_{43})^{k_{21}}3_{34}2_{23}(1_{12}2_{21})^{k_{11}}1_{12}}. \end{aligned} \tag{7.67}$$

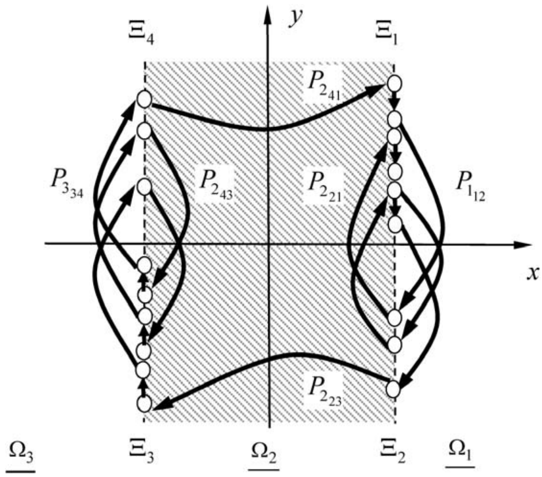


Figure 7.14. A mapping structure for a periodic motion.

From mapping structures of periodic motions, the switching sets for a specific regular motion can be determined through solving a set of nonlinear equations. For a periodic motion pertaining to mapping structure in Eq. (7.67)

$$P_{\substack{2_{41}(3_{34}2_{43})^{k_{2n}}3_{34}2_{23}(1_{12}2_{21})^{k_{1n}}1_{12}\cdots 2_{41}(3_{34}2_{43})^{k_{2m}}3_{34}2_{23}(1_{12}2_{21})^{k_{1m}}1_{12}\cdots\\ 2_{41}(3_{34}2_{43})^{k_{21}}3_{34}2_{23}(1_{12}2_{21})^{k_{11}}1_{12}}}\mathbf{y}_i = \mathbf{y}_{i+4n+2\sum_{m=1}^{n}(k_{1m}+k_{2m})} \tag{7.68}$$

where $\mathbf{y} = (\Omega t_i, y_i)^{\mathrm{T}}$. A set of vector equations is as

$$\begin{cases} \mathbf{f}^{(1_{12})}(\mathbf{y}_{i+1}, \mathbf{y}_i) = 0, \\ \mathbf{f}^{(2_{21})}(\mathbf{y}_{i+2}, \mathbf{y}_{i+1}) = 0, \\ \vdots \\ \mathbf{f}^{(2_{41})}(\mathbf{y}_{i+4n+2\sum_{m=1}^{n}(k_{1m}+k_{2m})}, \mathbf{y}_{i+4n+2\sum_{m=1}^{n}(k_{1m}+k_{2m})-1}) = 0 \end{cases} \tag{7.69}$$

where $\mathbf{f}^{(J_{\alpha\beta})} = (f_1^{(J_{\alpha\beta})}, f_2^{(J_{\alpha\beta})})^{\mathrm{T}}$ is relative to governing equations of mapping $P_{J_{\alpha\beta}}$ ($J_{\alpha\beta} = 1_{12}, 2_{21}, 2_{23}, 3_{34}, 2_{43}, 2_{41}$). The periodicity of the period-1 motion per N periods requires $\mathbf{y}_{i+4n+2\sum_{m=1}^{n}(k_{1m}+k_{2m})} = \mathbf{y}_i$,

$$(\Omega t_{i+4n+2\sum_{m=1}^{n}(k_{1m}+k_{2m})}, y_{i+4n+2\sum_{m=1}^{n}(k_{1m}+k_{2m})})^{\mathrm{T}} \equiv (\Omega t_i + 2N\pi, y_i)^{\mathrm{T}}. \tag{7.70}$$

Solving Eqs. (7.69) and (7.70) generates the switching sets for periodic motions. Once the switching sets of the periodic motion are obtained, the local stability and bifurcation should be discussed to determine the property of the periodic motion. The local stability and bifurcation for period-1 motion can be determined through the corresponding Jacobian matrix of the Poincaré mapping. From Eq. (7.68), the Jacobian matrix is computed by the chain rule, i.e.,

$$\begin{aligned} DP &= DP_{2_{41}(3_{34}2_{43})^{k_{2n}}3_{34}2_{23}(1_{12}2_{21})^{k_{1n}}1_{12}\cdots 2_{41}(3_{34}2_{43})^{k_{2m}}3_{34}2_{23}(1_{12}2_{21})^{k_{1m}}1_{12}\cdots 2_{41}(3_{34}2_{43})^{k_{21}}3_{34}2_{23}(1_{12}2_{21})^{k_{11}}1_{12}} \\ &= \prod_{m=1}^{n} \underbrace{(DP_{2_{41}} \cdot DP^{k_{2m}}_{3_{34}2_{43}} \cdot DP_{3_{34}} \cdot DP_{2_{23}} \cdot DP^{k_{1m}}_{1_{12}2_{21}} \cdot DP_{1_{12}})}_{m\text{th set}}. \end{aligned} \tag{7.71}$$

Two unknowns $(\Omega t_{i+1}, y_{i+1})$ is expressed through the initial conditions, i.e., $\varphi_{i+1} = \varphi_{i+1}(\varphi_i, y_i)$ and $y_{i+1} = y_{i+1}(\varphi_i, y_i)$ with $\varphi_i = \Omega t_i$. The eigenvalues of a fixed point for the periodic motion mapping are expressed through the trace (i.e., $\mathrm{Tr}(DP)$) and determinant ($\mathrm{Det}(DP)$) of the Jacobian matrix DP, i.e.,

$$\lambda_{1,2} = \frac{1}{2}\left[\mathrm{Tr}(DP) \pm \sqrt{\Delta}\right] \tag{7.72}$$

where $\Delta = [\mathrm{Tr}(DP)]^2 - 4\mathrm{Det}(DP)$. If $\Delta < 0$, Eq. (7.72) is expressed by

$$\lambda_{1,2} = \mathrm{Re}(\lambda) \pm j\mathrm{Im}(\lambda), \quad \text{with } j = \sqrt{-1}. \tag{7.73}$$

Notice that $\mathrm{Re}(\lambda) = \frac{1}{2}\mathrm{Tr}(DP)$ and $\mathrm{Im}(\lambda) = \frac{1}{2}\sqrt{|\Delta|}$. If two eigenvalues lie inside the unit circle, then the period-1 motion pertaining to the fixed point of the Poincaré mapping is stable. If one of them lie outside the unit circle, the periodic motion is unstable. Namely, the stable periodic motion requires the eigenvalues be

$$|\lambda_i| < 1 \quad (i = 1, 2). \tag{7.74}$$

When this condition is not satisfied, the periodic motion is unstable. For complex eigenvalues $|\lambda_{1,2}| < 1$, the periodic motion is a stable focus, while for $|\lambda_{1,2}| > 1$, the periodic motion becomes an unstable focus. If $|\lambda_{1 \text{ or } 2}| = 1$ with complex numbers, i.e.,

$$\mathrm{Det}(DP) = 1, \tag{7.75}$$

then the Neimark bifurcation occurs. For two eigenvalues being real, the stable-node periodic motion requires $\max_{i=1,2} |\lambda_i| < 1$ and the unstable node (or saddle) periodic motion requires $\max_{i=1,2} |\lambda_i| > 1$. The periodic motion with the first kind saddle-node requires $\lambda_{\alpha\in\{1,2\}} > 1$ and $|\lambda_{\beta\in\{1,2\}}| < 1$ $(\alpha \neq \beta)$. The saddle-node of the second kind needs $\lambda_{\alpha\in\{1,2\}} < -1$ and $|\lambda_{\beta\in\{1,2\}}| < 1$ $(\alpha \neq \beta)$. If one of the two eigenvalues is -1 (i.e., $\lambda_{1(\text{or }2)} = -1$) and the other one is inside the unit cycle, the period-doubling bifurcation occurs. Further, the period-doubling bifurcation condition is

$$\mathrm{Tr}(DP) + \mathrm{Det}(DP) + 1 = 0. \tag{7.76}$$

If one of the two eigenvalues is +1 (i.e., $\lambda_{1(\text{or }2)} = +1$) and the second one is inside the unit cycle, the first saddle-node bifurcation occurs. Similarly, the corresponding bifurcation condition is

$$\text{Det}(DP) + 1 = \text{Tr}(DP). \tag{7.77}$$

Periodic motions for this discontinuous system are illustrated. Consider parameters $m_{1,3} = 1$, $m_2 = 5$, $E_{1,2} = 1$, $Q_0 = 500$, $k_{1,3} = 2000$, $k_2 = 100$ and $b_{1,2,3} = 0.2$. The periodic motions with mapping $P_{2_{41}(3_{34}2_{43})^k 3_{34}2_{23}(1_{12}2_{21})^k 1_{12}}$ ($k = 0, 1, 2$) are predicted analytically. From the mapping structure of $P_{2_{41}(3_{34}2_{43})^k 3_{34}2_{23}(1_{12}2_{21})^k 1_{12}}$, the analytically predicted switching phases and velocities are illustrated in Figs. 7.15–7.17. This analytical prediction gives two branches of asymmetrical periodic motions instead of one branch of asymmetric periodic motion obtained in numerical simulations. The acronyms "SGB" and "AGB" represent the *symmetric* and *asymmetric* grazing bifurcations, respectively. The acronyms "SNB" and "PDB" are the saddle-node bifurcation of the first kind and the period-doubling bifurcation, respectively. The chain vertical lines are the location of grazing bifurcation. The dashed lines are the locations of the saddle-node bifurcations. The thick solid and dashed curves represent the stable and unstable, symmetric, periodic motions, respectively. The light solid and symbol curves are the stable and unstable, asymmetric periodic motions, respectively. The hollow and filled circular symbols are two different branches of the asymmetric motion.

The symmetric and asymmetric, periodic motion for $P_{2_{41}3_{34}2_{23}1_{12}}$ exists in $\Omega \in [3.32, 3.64] \cup [3.64, 6.88] \cup [6.882, 15.57]$, and $\Omega \in [5.72, 6.88]$, respectively. The stable and unstable, symmetric, periodic motions lie in $\Omega \in [3.32, 3.64] \cup [6.882, 15.57]$ and $\Omega \in [3.64, 6.88]$, accordingly. Before the two asymmetric motions become locally unstable, the grazing motion occurs. Therefore the frequency range $\Omega \in [5.72, 6.88]$ is only for the stable asymmetric motion. In Fig. 7.15, the saddle-node of the first kind occurs at the point from the symmetric to asymmetric motion. Once the grazing occurs, the periodic motion will vanish. The detailed discussion of the grazing bifurcation can be referred to Luo and Chen (2005a, 2005b, 2006). But the grazing bifurcations for symmetric and asymmetric periodic motion of mapping $P_{2_{41}3_{34}2_{23}1_{12}}$ occur at $\Omega \approx 3.32$ and $\Omega \approx 5.72$, respectively. The two asymmetric motions satisfy the symmetric property in Luo (2005e, 2005f). Such symmetry will be discussed in next chapter. When the grazing of the symmetric and asymmetric periodic motions occurs, the two periodic motions are still locally stable. This grazing caused the motion switching from the stable periodic motion into another motion. Two grazing bifurcations of the symmetric periodic motion concur at the two boundaries. However, the grazing for asymmetric motion only occurs at one boundary. The two different asymmetric motions have the different boundary locations for grazing, but the

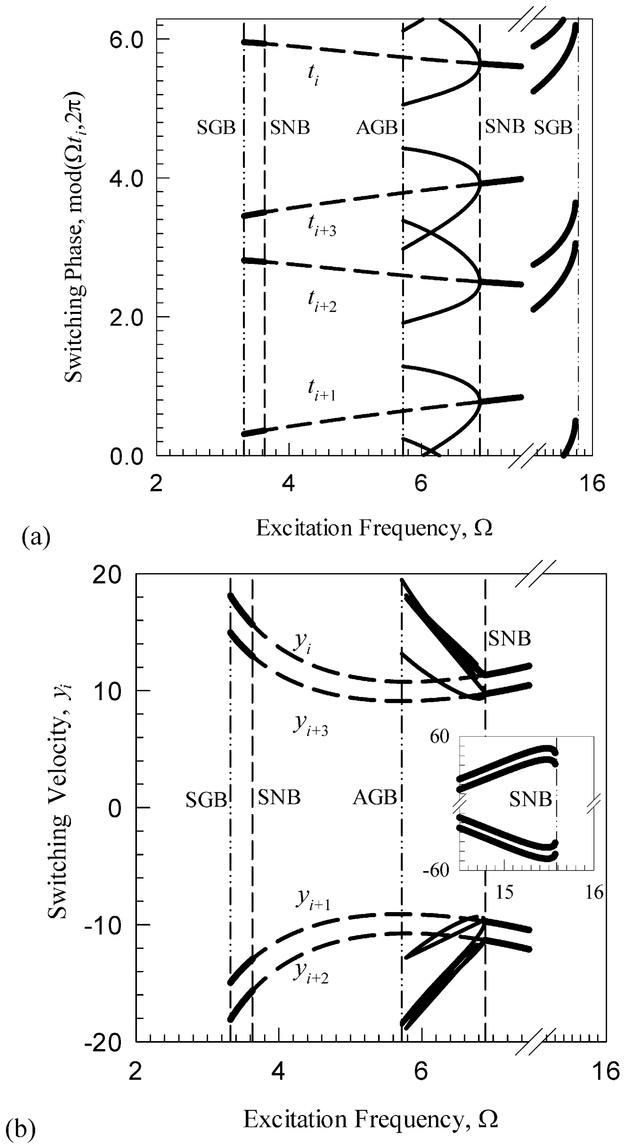


Figure 7.15. Analytical prediction of periodic motion $P_{2_{41}3_{34}2_{23}1_{12}}$: (a) switching phase, and (b) switching velocity varying with excitation frequency ($m_{1,3} = 1$, $m_2 = 5$, $E_{1,2} = 1$, $Q_0 = 500$, $k_{1,3} = 2000$, $k_2 = 100$, $b_{1,2,3} = 0.2$.)

two grazing bifurcations for the two asymmetric motions also satisfy the skew symmetry properties as in Luo (2005b, 2005c).

In Fig. 7.16, the symmetric and asymmetric periodic motions of $P_{241(334243)334223(112221)112}$ exist in ranges of $\Omega \in [2.15, 3.44] \cup [3.44, 3.97]$ and $\Omega \in [2.92, 3.31] \cup [3.31, 3.44]$, respectively. The stable periodic motions lie in $\Omega \in [3.31, 3.44] \cup [3.44, 3.97]$, but the unstable motions are in $\Omega \in [2.15, 3.44] \cup [2.92, 3.31]$. The grazing of the periodic motion occurs at the unstable periodic motion. The grazing bifurcation for the symmetric motion is at $\Omega = 2.15$ and 3.97. However, the grazing frequency for the asymmetric motion is $\Omega \approx 2.92$. For the asymmetric motion, the period-doubling bifurcation is at $\Omega \approx 3.31$. In Fig. 7.17, consider more complex periodic motion with the mapping $P_{241(334243)^2 334223(112221)^2 112}$. The corresponding symmetric and asymmetric, periodic motions exist in $\Omega \in [1.64, 2.34] \cup [2.34, 2.49]$ and $[2.01, 2.29] \cup [2.29, 2.34]$, respectively. Grazing and local stability behaviors are the same as the ones of $P_{241(334243)334223(112221)112}$. The grazing of the symmetric periodic motion is at $\Omega \approx \{1.64, 2.49\}$. However, for the asymmetric periodic motion, the grazing frequency is $\Omega \approx 2.01$ and period-doubling frequency is $\Omega \approx 2.29$. Based on this technique, all possible periodic motions including stable and stable ones can be obtained.

From the analytical prediction, illustrations of the complicated periodic motion are very interesting. Consider the same parameters ($m_{1,3} = 1, m_2 = 5, E_{1,2} = 1, Q_0 = 500, k_{1,3} = 2000, k_2 = 100$ and $b_{1,2,3} = 0.2$) again. The phase trajectories and displacement responses of the periodic motions relative to the mapping structures $P_{241(334243)334223(112221)112}$ and $P_{241(334243)^2 334223(112221)^2 112}$ are illustrated in Figs. 7.18–7.23. In Figs. 7.18 and 7.19, the symmetric, periodic motions relative to $P_{241(334243)334223(112221)112}$ and $P_{241(334243)^2 334223(112221)^2 112}$ are presented, respectively. The initial condition and excitation frequency for the symmetric periodic motion of $P_{241(334243)334223(112221)112}$ are $\Omega = 3.6$, $\Omega t_i \approx 6.0850$, $x_i = 1$, $y_i \approx -6.2583$. However, for mapping $P_{241(334243)^2 334223(112221)^2 112}$, the initial condition is $\Omega = 2.35$, $\Omega t_i \approx 5.2385$, $x_i = 1$, $y_i \approx 8.2854$. The symmetry of the motion in phase plane is clearly observed. The velocity discontinuity caused by the impact is clearly observed as well. The two stable, asymmetric, periodic motions pertaining to $P_{241(334243)334223(112221)112}$ ($\Omega = 3.35$ and $x_i = 1$) are illustrated in Figs. 7.20 and 7.21, accordingly. The lower asymmetric motion is based on the initial condition $\Omega t_i \approx 5.2270$, $y_i \approx 9.6488$. However, the upper asymmetric motion is simulated via the initial condition $\Omega t_i \approx 5.5227$, $y_i \approx 10.3250$. Compared to the symmetric motion, the motions relative to P_{223} and P_{241} are asymmetric. But the two asymmetric motions are skew symmetric because the mapping structure is skew symmetric. Two stable asymmetric periodic motions pertaining to $P_{241(334243)^2 334223(112221)^2 112}$ ($\Omega = 3.2$ and $x_i = 1$) are illustrated in Figs. 7.22 and 7.23 as well. The lower asymmetric motion is based on the initial conditions $\Omega t_i \approx 5.1670$, $y_i \approx 8.3312$, and the initial conditions for the upper

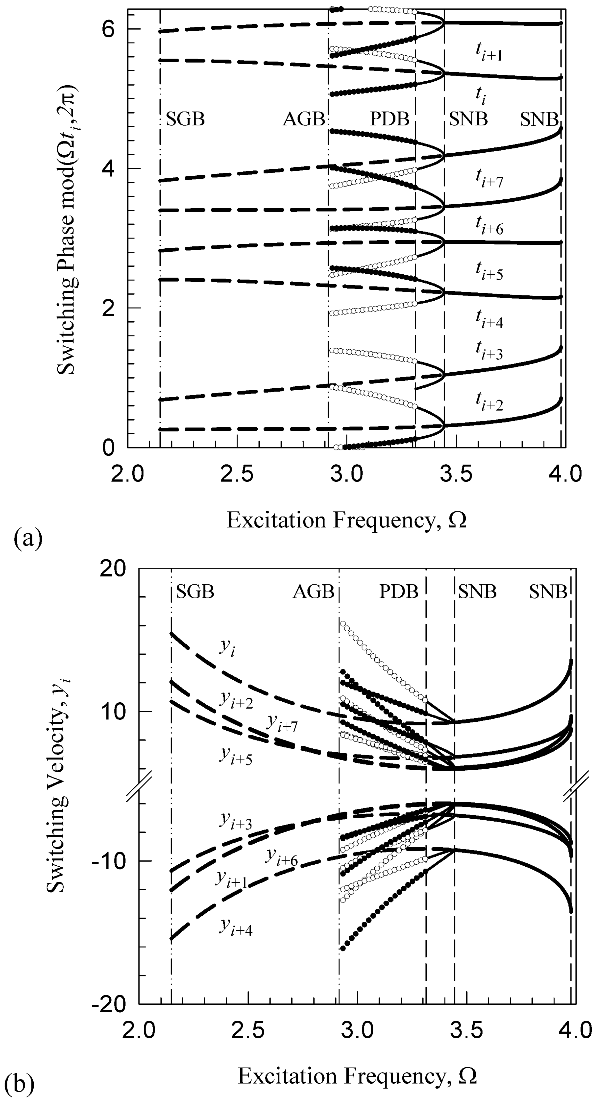


Figure 7.16. Analytical prediction of periodic motion $P_{2_{41}(3_{34}2_{43})3_{34}2_{23}(1_{12}2_{21})1_{12}}$: (a) switching phase, and (b) switching velocity varying with excitation frequency ($m_{1,3} = 1, m_2 = 5, E_{1,2} = 1, Q_0 = 500, k_{1,3} = 2000, k_2 = 100, b_{1,2,3} = 0.2$.)

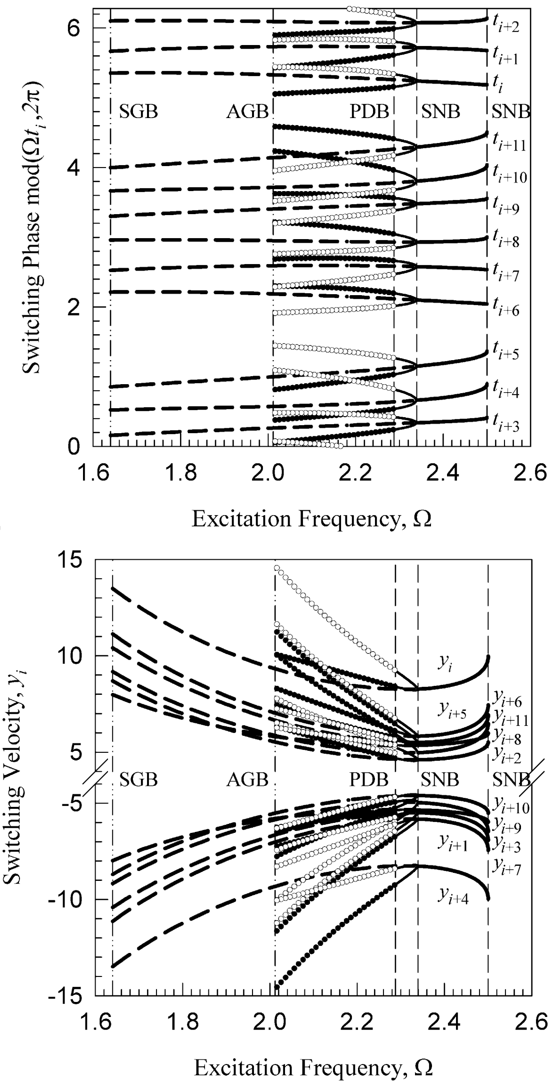


Figure 7.17. Analytical prediction of periodic motion $P_{2_{41}(3_{34}2_{43})^2 3_{34}2_{23}(1_{12}2_{21})^2 1_{12}}$: (a) switching phase, and (b) switching velocity varying with excitation frequency ($m_{1,3} = 1$, $m_2 = 5$, $E_{1,2} = 1$, $Q_0 = 500$, $k_{1,3} = 2000$, $k_2 = 100$, $b_{1,2,3} = 0.2$.)

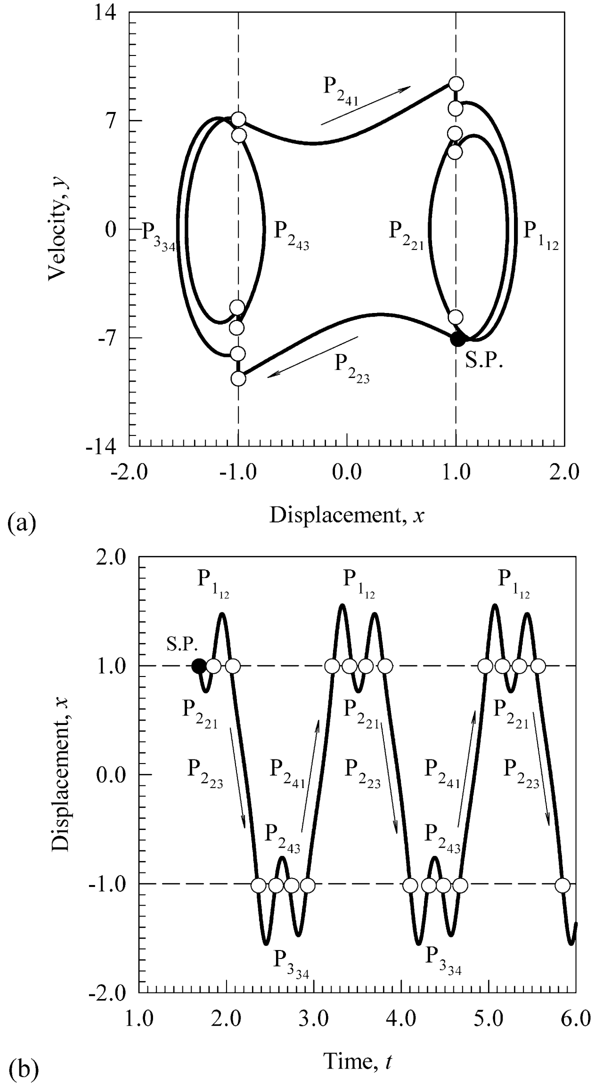


Figure 7.18. (a) Phase trajectory, and (b) displacement response of the symmetric periodic motions $P_{2_{41}(3_{34}2_{43})3_{34}2_{23}(1_{12}2_{21})1_{12}}$ with initial conditions $\Omega = 3.6$, $\Omega t_i \approx 6.0850$, $x_i = 1$, $y_i \approx -6.2583$. ($m_{1,3} = 1, m_2 = 5$, $E_{1,2} = 1$, $Q_0 = 500$, $k_{1,3} = 2000$, $k_2 = 100$, $b_{1,2,3} = 0.2$.)

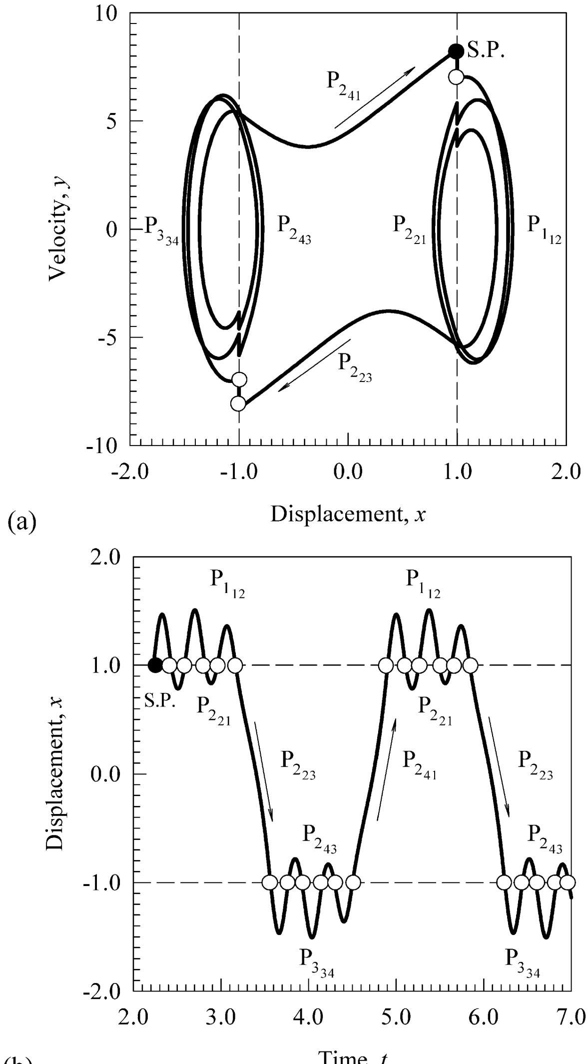
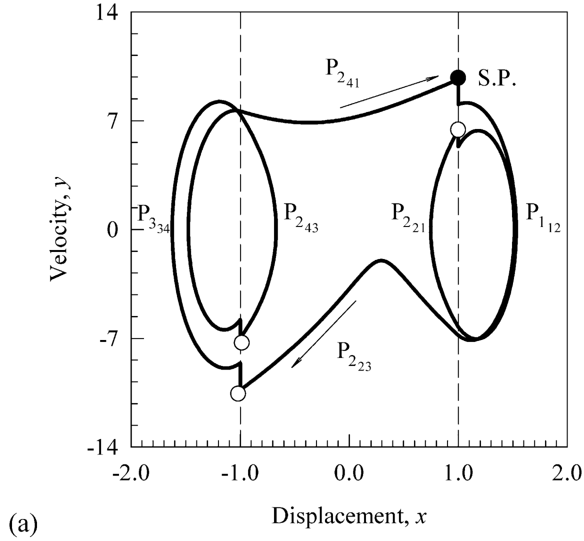


Figure 7.19. (a) Phase trajectories, and (b) displacement responses of the stable symmetric periodic motions of $P_{2_{41}(3_{34}2_{43})^2 3_{34}2_{23}(1_{12}2_{21})^2 1_{12}}$ with initial conditions $\Omega = 2.35$, $\Omega t_i \approx 5.2385$, $x_i = 1, y_i \approx 8.2854$. ($m_{1,3} = 1$, $m_2 = 5$, $E_{1,2} = 1$, $Q_0 = 500$, $k_{1,3} = 2000$, $k_2 = 100$, $b_{1,2,3} = 0.2$.)



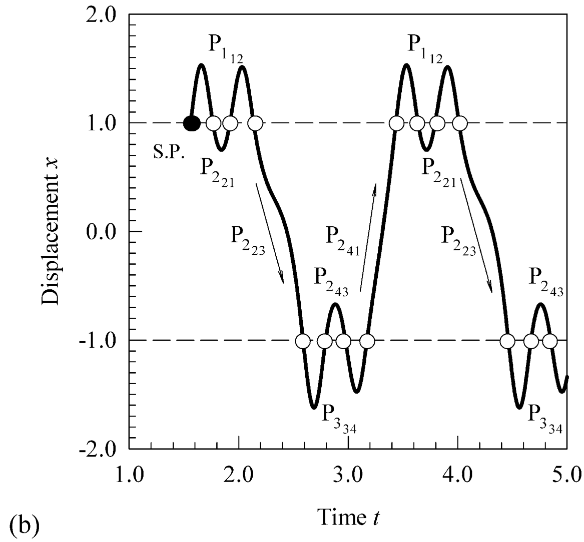


Figure 7.20. (a) Phase trajectories, and (b) displacement responses for the lower asymmetric periodic motion pertaining to $P_{2_{41}(3_{34}2_{43})3_{34}2_{23}(1_{12}2_{21})1_{12}}$ ($\Omega = 3.35$ and $x_i = 1$) with $\Omega t_i \approx 5.2270$, $y_i \approx 9.6488$. ($m_{1,3} = 1, m_2 = 5, E_{1,2} = 1, Q_0 = 500, k_{1,3} = 2000, k_2 = 100, b_{1,2,3} = 0.2$.)

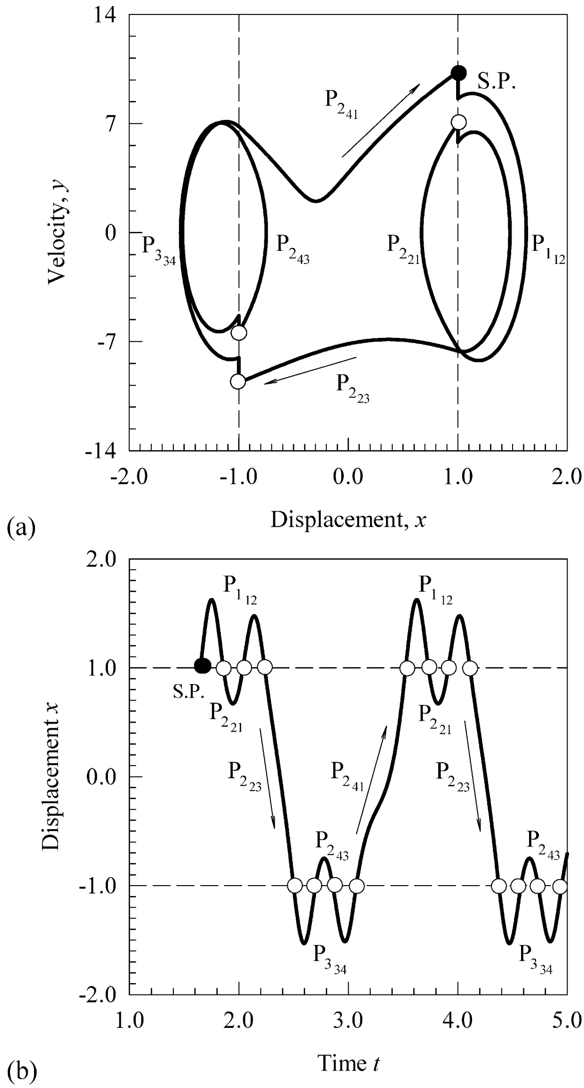
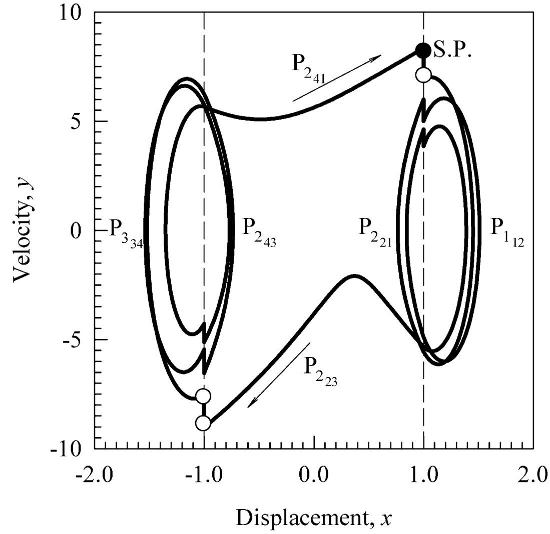
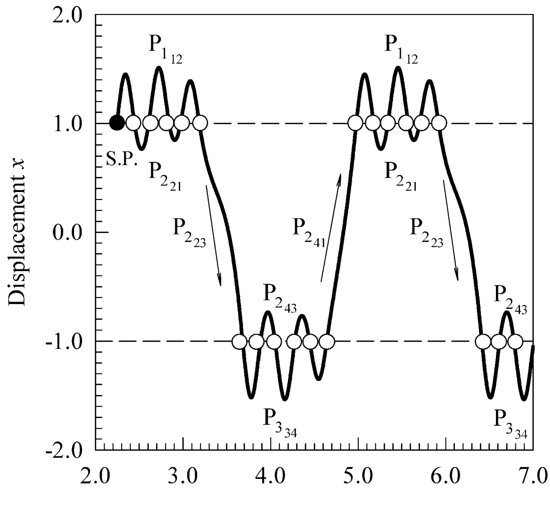


Figure 7.21. (a) Phase trajectory, and (b) displacement response for the upper asymmetric periodic motions pertaining to $P_{2_{41}(3_{34}2_{43})3_{34}2_{23}(1_{12}2_{21})1_{12}}$ ($\Omega = 3.35$ and $x_i = 1$) with $\Omega t_i \approx 5.5227$, $y_i \approx 10.3250$. ($m_{1,3} = 1, m_2 = 5$, $Q_0 = 500$, $k_{1,3} = 2000$, $k_2 = 100$, $b_{1,2,3} = 0.2$.)

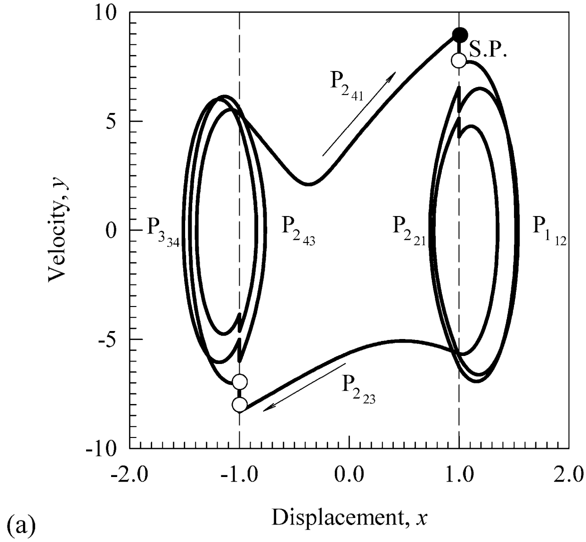


(a)



(b)

Figure 7.22. (a) Phase trajectory, and (b) displacement response for the lower asymmetric periodic motion pertaining to $P_{2_{41}(3_{34}2_{43})^2 3_{34}2_{23}(1_{12}2_{21})^2 1_{12}}$ ($\Omega = 2.3$ and $x_i = 1$) with $\Omega t_i \approx 5.1670$, $y_i \approx 8.3312$. ($m_{1,3} = 1, m_2 = 5, E_{1,2} = 1, Q_0 = 500, k_{1,3} = 2000, k_2 = 100, b_{1,2,3} = 0.2$.)



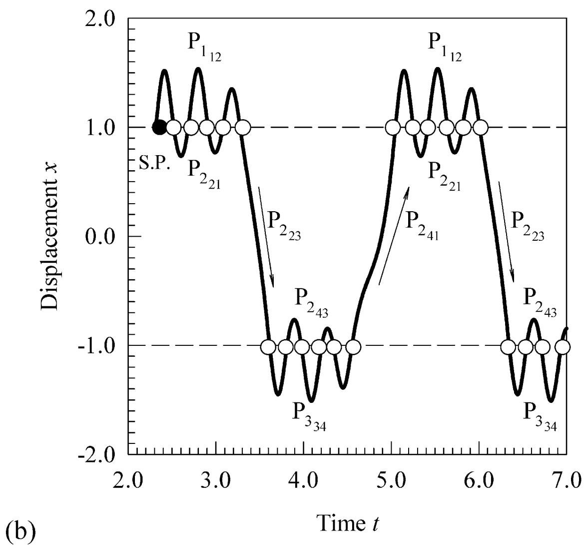


Figure 7.23. (a) Phase trajectory, and (b) displacement response for the upper asymmetric periodic motion pertaining to $P_{2_{41}(3_{34}2_{43})^2 3_{34}2_{23}(1_{12}2_{21})^2 1_{12}}$ ($\Omega = 2.3$ and $x_i = 1$) with $\Omega t_i \approx 5.3255$, $y_i \approx 9.0293$. ($m_{1,3} = 1, m_2 = 5, E_{1,2} = 1, Q_0 = 500, k_{1,3} = 2000, k_2 = 100, b_{1,2,3} = 0.2$.)

asymmetric motion are $\Omega t_i \approx 5.3255$, $y_i \approx 9.0293$. The solution symmetry of the periodic motions will satisfy the conditions in Luo (2005e). Once one of the two asymmetric periodic motions is known, the other asymmetric periodic motion can be obtained. The detailed discussion can be referred to Luo (2005b). If readers are more interested in determining the complicated periodic motions in other discontinuous dynamical systems through the mapping structure technique, one may refer to Luo and coworkers' papers (e.g., Han et al., 1995; Luo, 2002; Luo, 2005e; Luo and Gegg, 2006a, 2006b; Luo and Zwiegart Jr., 2005).

From this section, the switching sets and domains are named separately and arbitrarily. The domain is used for the primary index for mapping and the switching sets are used for the second index for mapping. This naming system gives a very complicated notation. For simple discontinuous systems, such a naming of the switching sets and domain is not necessarily to be adopted. However, for complicated discontinuous dynamical systems, this naming system should be used to identify the possible mappings. The mapping dynamics of periodic motion in any system is to develop the mapping relations from which expected periodic motions can be analytically predicted. The mapping dynamics provides a possibility to obtain all stable and unstable periodic motions in dynamical system rather than only one of stable solutions given by numerical simulations.

Chapter 8

Symmetry and Fragmentized Strange Attractor

In Chapter 7, mapping dynamics in discontinuous systems provided an analytical prediction of all stable and unstable periodic flows rather than only one of stable solutions given by numerical simulations. The symmetry of periodic flows in discontinuous systems was observed. This chapter will focus on the symmetry of steady-state flows in discontinuous dynamical systems. The grazing mapping clusters will be introduced. Based on the grazing mapping clusters, the mapping structures are used to discuss the solution symmetry property of periodic and chaotic motions in discontinuous dynamical systems. The initial and final switching manifolds are introduced. The fragmentation phenomena of the strange attractors of chaotic motion in discontinuous dynamic systems will be described. The grazing mechanism of the strange attractor fragmentation will be discussed. The fragmentation of the strange attractors extensively exists in discontinuous dynamical systems, which will help one better understand chaotic motions in discontinuous dynamical systems.

8.1. Symmetric discontinuity

In this chapter, the naming system of Chapter 7 will not be adopted. As in Chapter 2, discontinuous dynamic systems with symmetry are stated. Consider an n-dimensional dynamic system consisting of m-subsystems on m-accessible subdomains Ω_p $(p = 1, 2, \ldots, m)$ in a domain $\subset \Re^n$. With the inaccessible domain Ω_0, the universal domain is expressed by $\mho = \bigcup_{p=1}^{m} \Omega_p \cup \Omega_0$. The m-accessible domain and inaccessible domain Ω_0 are separated by boundaries $\partial\Omega_{p_1 p_2} \subset \Re^{n-1}$ $(p_1, p_2 \in \{0, 1, 2, \ldots, m\})$ in phase space determined by a function $\varphi_{p_1 p_2}(\mathbf{x}, t) = 0$. Among all the boundaries, there are some pairs of boundaries $\partial\Omega_{p_1 p_2}$ with symmetry. For the pth accessible domain, there is a continuous system in the form of

$$\dot{\mathbf{x}} = \mathbf{f}^{(p)}(\mathbf{x}, \boldsymbol{\mu}_p) + \mathbf{g}(\mathbf{x}, \varphi, \boldsymbol{\pi}), \quad \mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \Omega_p \tag{8.1}$$

where the forcing vector functions $\mathbf{g} = (g_1, g_2, \ldots, g_n)^{\mathrm{T}}$ are bounded periodic functions with period $T = 2\pi/\Omega$. The forcing frequency is Ω. The phase

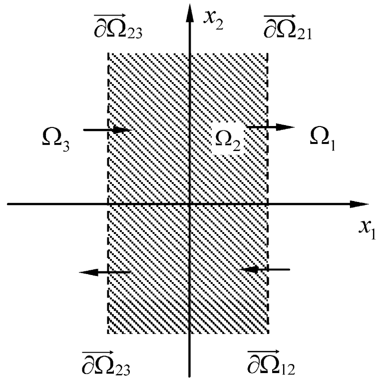


Figure 8.1. Two symmetric domains in phase plane.

variable is $\varphi = \Omega t$ and the parameter vector $\boldsymbol{\pi} = (\pi_1, \pi_2 \ldots, \pi_{m_1})^{\mathrm{T}}$. The vector functions $\mathbf{f}^{(p)} = (f_1^{(p)}, f_2^{(p)}, \ldots, f_n^{(p)})^{\mathrm{T}}$ with system parameter vector $\boldsymbol{\mu}_p = (\mu_1^{(p)}, \mu_2^{(p)}, \ldots, \mu_{m_2}^{(p)})^{\mathrm{T}}$ are C^r-continuous ($r \geqslant 2$). In all accessible subdomains Ω_p ($p = 1, 2, \ldots, m$), the dynamical system in Eq. (8.1) is continuous and there is a continuous flow expressed by

$$\mathbf{x}^{(p)}(t) = \boldsymbol{\Phi}^{(p)}\big(\mathbf{x}^{(p)}(t_0), t, \boldsymbol{\mu}_p, \boldsymbol{\pi}\big),$$
$$\mathbf{x}^{(p)}(t_0) = \boldsymbol{\Phi}^{(p)}\big(\mathbf{x}^{(p)}(t_0), t_0, \boldsymbol{\mu}_p, \boldsymbol{\pi}\big). \tag{8.2}$$

Consider one of the simplest sub-accessible domains in a two-dimensional dynamical system in Fig. 8.1. The universal domain is formed by three accessible subdomains Ω_p ($p = 1, 2, 3$). With oriented directions, two boundaries $\partial\Omega_{p_1p_2}$ ($p_1, p_2 \in \{1, 2, 3\}$) exist. The accessible subdomains (Ω_1 and Ω_3) are of the skew-symmetry. The direction-oriented boundaries ($\overrightarrow{\partial\Omega}_{21}, \overrightarrow{\partial\Omega}_{12}$) and ($\overrightarrow{\partial\Omega}_{23}, \overrightarrow{\partial\Omega}_{32}$) are of the skew-symmetry, respectively. On the corresponding symmetric domains, the dynamical systems in Eq. (8.1) are of the skew symmetry. On the boundaries ($\overrightarrow{\partial\Omega}_{12}$ and $\overrightarrow{\partial\Omega}_{21}$), the governing equation is defined through $\varphi_{12}(\mathbf{x}, t) = \varphi_{21}(\mathbf{x}, t) = x_1 - E = 0$, and on the corresponding symmetric boundaries ($\overrightarrow{\partial\Omega}_{23}$ and $\overrightarrow{\partial\Omega}_{32}$), the governing equation is determined by $\varphi_{23}(\mathbf{x}, t) = \varphi_{32}(\mathbf{x}, t) \equiv x_1 + E = 0$.

From the above discussion, besides the assumptions in Chapter 2, the extra conditions should be added to restrict our discussion. In this chapter, the following assumptions will be considered:

(A1) The system in Eq. (8.1) possesses time-continuity.
(A2) For an unbounded domain Ω_p, there is an open domain $D_p \subset \Omega_p$. On the domain D_p, the vector field and the corresponding flow are bounded for

$t \in [0, \infty)$, i.e.,

$$\left\|\mathbf{f}^{(p)}\right\| + \|\mathbf{g}\| \leqslant K_1 \text{ (const)} \quad \text{and} \quad \left\|\boldsymbol{\Phi}^{(p)}\right\| \leqslant K_2 \text{ (const)}. \tag{8.3}$$

(A3) For a bounded domain Ω_p, there is an open domain $D_p \subset \Omega_p$. On the domain D_p, the vector field and the corresponding flow are bounded for $t \in [0, \infty)$, i.e.,

$$\left\|\mathbf{f}^{(p)}\right\| + \|\mathbf{g}\| \leqslant K_1 \text{ (const)} \quad \text{and} \quad \left\|\boldsymbol{\Phi}^{(p)}\right\| < \infty. \tag{8.4}$$

(A4) The dynamical subsystems possess symmetry at least in two corresponding symmetric domains.

(A5) The flow on the discontinuous boundaries is C^1-discontinuous or there is a transport law to connect two flows in two different domains.

8.2. Switching sets and mappings

For convenience, without loss of generality, the switching sets are from the boundary $\overrightarrow{\partial\Omega}_{p_1 p_2}$ $(q = 0, 1, 2, \ldots, M)$:

$$\Xi_q = \left\{\left(\operatorname{mod}(\Omega t_i, 2\pi), \mathbf{x}_i\right) \mid \varphi_{p_1 p_2}(t_i, \mathbf{x}_i) = 0,\ p_1, p_2 \in \{0, 1, 2, \ldots, m\}\right\} \subseteq \overrightarrow{\partial\Omega}_{p_1 p_2}. \tag{8.5}$$

From the concepts of gluing singular sets, the singular sets on the switching sets are stated as follows:

DEFINITION 8.1. For a discontinuous dynamical system in Eq. (8.1), a set on the boundary $\partial\Omega_{p_1 p_2}$ for $p_1, p_2 \in \{1, 2, \ldots, m\}$ is termed as

(i) the singular set if

$$\Gamma_{p_1 p_2} = \Big\{\left(\operatorname{mod}(\Omega t_i, 2\pi), \mathbf{x}_i^{(0)}\right) \mid \mathbf{x}_i^{(0)} \in \partial\Omega_{p_1 p_2},\ \mathbf{x} \in \Omega_\alpha, \lim_{\mathbf{x} \to \mathbf{x}_i^{(0)}} \left(\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{p_1 p_2}} \cdot \dot{\mathbf{x}}\right) = 0, \alpha \in \{p_1, p_2\},\ i \in \mathbb{N}\Big\}; \tag{8.6}$$

(ii) the input/output semi-singular set if

$$\Gamma^{(\alpha)}_{p_1 p_2} = \Big\{\left(\operatorname{mod}(\Omega t_i, 2\pi), \mathbf{x}_i^{(0)}\right) \mid \mathbf{x}_i^{(0)} \in \partial\Omega_{p_1 p_2},\ \mathbf{x} \in \Omega_\alpha, \lim_{\mathbf{x} \to \mathbf{x}_i^{(0)}} \left(\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{p_1 p_2}} \cdot \dot{\mathbf{x}}\right) = 0, \alpha = \{p_1 \text{ or } p_2\},\ i \in \mathbb{N}\Big\} \subset \Gamma_{p_1 p_2}; \tag{8.7}$$

(iii) the full-singular set if

$$\Gamma_{\overleftrightarrow{p_1 p_2}} = \Big\{ \big(\mathrm{mod}(\Omega t_i, 2\pi), \mathbf{x}_i^{(0)}\big) \mid \mathbf{x}_i^{(0)} \in \partial\Omega_{p_1 p_2},\ \mathbf{x} \in \Omega_\alpha, \lim_{\mathbf{x}\to\mathbf{x}_i^{(0)}} \big(\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{p_1 p_2}} \cdot \dot{\mathbf{x}}\big) = 0, \alpha = \{p_1 \text{ and } p_2\},\ i \in \mathbb{N} \Big\} \subset \Gamma_{p_1 p_2}. \tag{8.8}$$

A switching set on the discontinuous boundary $\partial\Omega_{p_1 p_2}$ is composed of two neighbored boundaries $\Xi_{q_1} \subseteq \overrightarrow{\partial\Omega}_{p_1 p_2}$ and $\Xi_{q_2} \subseteq \overleftarrow{\partial\Omega}_{p_1 p_2}$ connected by a singular point $\Gamma_{p_1 p_2}$, i.e.,

$$\Xi_{q_1 q_2} = \Xi_{q_1} \cup \Xi_{q_2} \cup \Gamma_{p_1 p_2} \subseteq \partial\Omega_{p_1 p_2} \tag{8.9}$$

for $q_1, q_2 \in \{0, 1, 2, \ldots, M\}$. The local mappings near the switching sets $\Xi_{q_1 q_2}$ are defined as

$$P_{J_1} : \Xi_{q_1} \to \Xi_{q_2}, \quad P_{J_2} : \Xi_{q_2} \to \Xi_{q_1} \tag{8.10}$$

for specified $J_1, J_2 \in \{1, 2, \ldots, N\}$. If $\Xi_{q_1} \subseteq \partial\Omega_{p_1 p_2}$ and $\Xi_{q_3} \not\subset \partial\Omega_{p_1 p_2}$, the global mapping in an accessible domain Ω_α ($\alpha \in \{p_1, p_2\}$) is defined as

$$P_{J_3} : \Xi_{q_1} \to \Xi_{q_3} \tag{8.11}$$

for specified $J_3 \in \{1, 2, \ldots, N\}$. On the switching sets $\Xi_{q_1} \subseteq \partial\Omega_{p_1 p_2}$, the sliding mapping is defined as

$$P_{J_4} : \Xi_{q_1} \to \Xi_{q_1} \tag{8.12}$$

for specified $J_4 \in \{1, 2, \ldots, N\}$. For a C^0-discontinuous switching set $\Xi_{q_4} \subseteq \partial\Omega_{p_1 p_2}$, there must be a transport law to maps it to another switching set, $\Xi_{q_5} \subset \partial\Omega_{p_3 p_4}$. Therefore, the transport mapping is defined as

$$P_{J_5} : \Xi_{q_4} \to \Xi_{q_5} \tag{8.13}$$

for specified $J_5 \in \{1, 2, \ldots, N\}$.

The switching sets, the local, global, sliding and transport mappings are depicted in Fig. 8.2. The forbidden zone is shown for permanently-nonpassable boundary. Consider a discontinuous dynamical system with domains given in Fig. 8.1 to show how to use the above concept. Four switching sets are defined as

$$\begin{aligned} \Xi_1 &= \big\{ \big(\mathrm{mod}(\Omega t_i, 2\pi), \mathbf{x}_i\big) \mid x_2 > 0, \varphi_{21}(\mathbf{x}_i, t) = x_1 - E = 0 \big\} \\ &\subseteq \overrightarrow{\partial\Omega}_{21}, \\ \Xi_2 &= \big\{ \big(\mathrm{mod}(\Omega t_i, 2\pi), \mathbf{x}_i\big) \mid x_2 < 0, \varphi_{12}(\mathbf{x}_i, t) = x_1 - E = 0 \big\} \\ &\subseteq \overrightarrow{\partial\Omega}_{12}, \end{aligned} \tag{8.14}$$

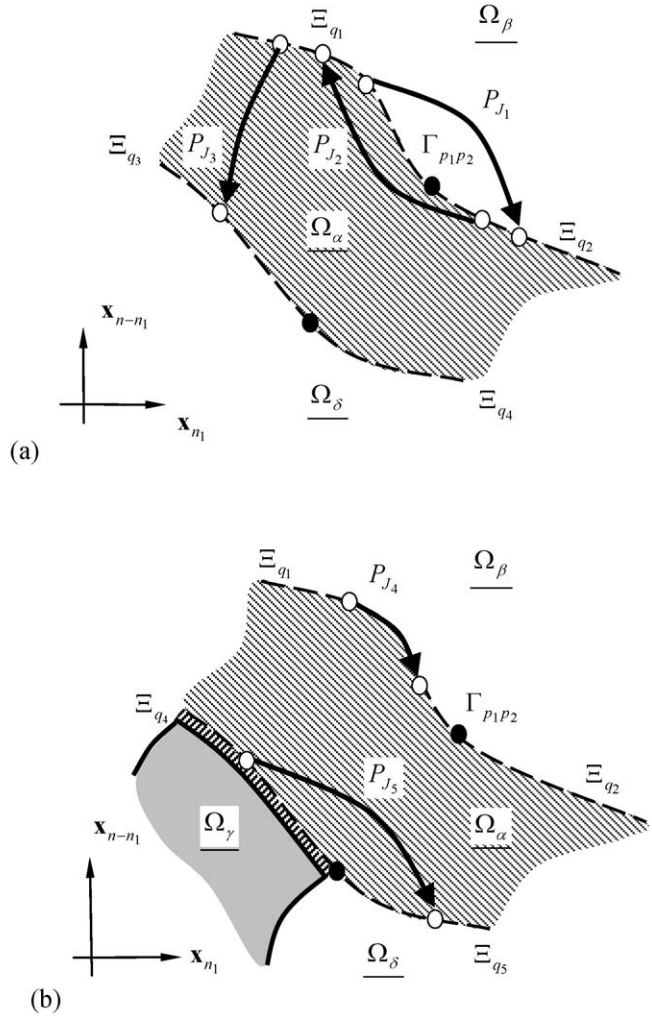


Figure 8.2. Switching sets and generic mappings: (a) local and global mappings, (b) sliding mapping and transport mapping. The shaded domain Ω_α is accessible, and the inaccessible domain is Ω_γ. The domain Ω_β is accessible and Ω_δ can be accessible or inaccessible.

$$
\begin{aligned}
\Xi_3 &= \big\{\big(\mathrm{mod}(\Omega t_i, 2\pi), \mathbf{x}_i\big) \mid x_2 < 0, \varphi_{23}(\mathbf{x}_i, t) = x_1 + E = 0\big\} \\
&\subseteq \overrightarrow{\partial\Omega}_{23}, \\
\Xi_4 &= \big\{\big(\mathrm{mod}(\Omega t_i, 2\pi), \mathbf{x}_i\big) \mid x_2 > 0, \varphi_{32}(\mathbf{x}_i, t) = x_1 - E = 0\big\} \\
&\subseteq \overrightarrow{\partial\Omega}_{32}
\end{aligned}
$$

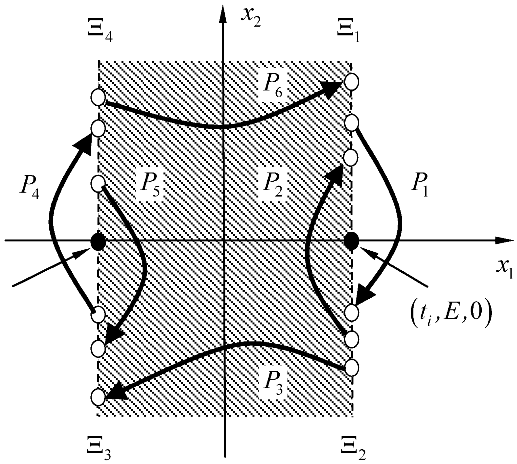


Figure 8.3. Switching sets and generic mappings for $x_1 = \pm E$.

and two singular points are

$$\begin{aligned}
\Gamma_{\overleftrightarrow{12}} &= \left\{ \left(\mathrm{mod}(\Omega t_i, 2\pi), \mathbf{x}_i \right) \mid x_2 = 0, \varphi_{12} = x_1 - E = 0 \right\} \subset \Gamma_{12}, \\
\Gamma_{\overleftrightarrow{23}} &= \left\{ \left(\mathrm{mod}(\Omega t_i, 2\pi), \mathbf{x}_i \right) \mid x_2 = 0, \varphi_{23} = x_1 - E = 0 \right\} \subset \Gamma_{23}.
\end{aligned} \tag{8.15}$$

The two sets are decomposed as

$$\begin{aligned}
\Xi_{12} &= \Xi_1 \cup \Xi_2 \cup \Gamma_{12} \subseteq \partial\Omega_{12}, \\
\Xi_{34} &= \Xi_3 \cup \Xi_4 \cup \Gamma_{23} \subseteq \partial\Omega_{23}.
\end{aligned} \tag{8.16}$$

From four subsets, the local mappings are

$$\begin{aligned}
&P_1 : \Xi_1 \to \Xi_2 \quad \text{and} \quad P_2 : \Xi_1 \to \Xi_2 \quad \text{on } \partial\Omega_{12}, \\
&P_4 : \Xi_3 \to \Xi_4 \quad \text{and} \quad P_5 : \Xi_4 \to \Xi_3 \quad \text{on } \partial\Omega_{23}
\end{aligned} \tag{8.17}$$

and the global mappings from $\partial\Omega_{12}$ and $\partial\Omega_{23}$ are

$$P_3 : \Xi_2 \to \Xi_3 \quad \text{and} \quad P_6 : \Xi_4 \to \Xi_1. \tag{8.18}$$

In this discontinuous system, no sliding mapping exists on the C^1-discontinuous boundaries. From the definition of mappings, the mappings P_J $(J = 1, 2, 4, 5)$ relative to one switching section are called the *local* mappings, and the mappings P_J $(J = 3, 6)$ relative to two switching sections are called the *global* mappings. The global mapping maps the motion from one switching boundary into another switching boundary. The local mapping is the self-mapping in the corresponding switching section. Six generic mappings are illustrated in Fig. 8.3.

8.3. Grazing and mappings symmetry

The initial and final times (t_i and t_{i+1}) are used for all the mappings P_J ($J = 1, 2, \ldots, N$) defined through in Eqs. (8.10)–(8.13), and the corresponding phases are $\varphi_i = \Omega t_i$ and $\varphi_{i+1} = \Omega t_{i+1}$. Equation (8.2) gives

$$\begin{aligned} &\mathbf{x}^{(p)}(t_{i+1}) = \boldsymbol{\Phi}^{(p)}\big(t_{i+1}, \mathbf{x}^{(p)}(t_i), t_i, \boldsymbol{\mu}_p, \boldsymbol{\pi}\big), \quad \text{or} \\ &\mathbf{x}^{(p)}(\varphi_{i+1}) = \boldsymbol{\Phi}_1^{(p)}\big(\varphi_{i+1}, \mathbf{x}^{(p)}(\varphi_i), \varphi_i, \boldsymbol{\mu}_p, \boldsymbol{\pi}\big). \end{aligned} \tag{8.19}$$

In an accessible domain Ω_p, the foregoing equations with boundary constraint equations give the governing equations for mapping P_J. Consider a notation $\mathbf{y}_i \equiv (\varphi_i, \mathbf{x}_i)^{\mathrm{T}} \in \Re^{n+1}$, the governing equations for mapping P_J ($J = 1, 2, \ldots, N$) with mapping relation $\mathbf{y}_{i+1} = P_J \mathbf{y}_i$ are

$$\mathbf{F}^{(J)}(\varphi_i, \mathbf{x}_i, \varphi_{i+1}, \mathbf{x}_{i+1}, \boldsymbol{\mu}_p, \boldsymbol{\pi}) = 0 \tag{8.20}$$

where $\mathbf{F}^{(J)} \in \Re^{n+1}$.

For all mappings P_J ($J = 1, 2, \ldots, N$), there are $2n_1$ symmetric mappings P_q ($q = 1, 2, \ldots, 2n_1$). The first n_1 mappings P_{q_1} ($q_1 = 1, 2, \ldots, n_1$) are symmetric with the last n_1 mappings P_{q_2} ($q_2 = n_1 + 1, n_1 + 2, \ldots, 2n_1$), respectively. From Assumption (A4), the corresponding subsystems in domain $\Omega_{p_{q_1}}$ and $\Omega_{p_{q_2}}$ ($p_{q_1}, p_{q_2} \in \{1, 2, \ldots, m\}$) are symmetric. Therefore, $\boldsymbol{\mu}_{p_{q_1}} = \boldsymbol{\mu}_{p_{q_2}}$. For convenience, let us define

$$\begin{aligned} &\hat{q} = \mathrm{mod}(q + n_1 - 1, 2n_1) + 1, \\ &\hat{\varphi}_i^{(q)} = \mathrm{mod}\big(\varphi_i^{(q)}, 2(2M_1 + 1)\pi\big), \\ &\hat{\varphi}_i^{(\hat{q})} = \mathrm{mod}\big(\varphi_i^{(\hat{q})}, 2(2M_1 + 1)\pi\big). \end{aligned} \tag{8.21}$$

DEFINITION 8.2. For a discontinuous dynamical system in Eq. (8.1), under a transformation $T_P : P_q \to P_{\hat{q}}$ during $(2M_1 + 1)$-periods with

$$\begin{aligned} &\hat{\varphi}_i^{(q)} = \mathrm{mod}\big((2M_1 + 1)\pi + \hat{\varphi}_i^{(\hat{q})}, 2(2M_1 + 1)\pi\big), \\ &\mathbf{x}_i^{(\hat{q})} = -\mathbf{x}_i^{(q)} \end{aligned} \tag{8.22}$$

if a relation

$$\begin{aligned} &\mathbf{F}^{(\hat{q})}\big(\varphi_i^{(\hat{q})}, \mathbf{x}_i^{(\hat{q})}, \varphi_{i+1}^{(\hat{q})}, \mathbf{x}_{i+1}^{(\hat{q})}, \boldsymbol{\mu}_{p_q}, \boldsymbol{\pi}\big) \\ &\quad = -\mathbf{F}^{(q)}\big(\varphi_i^{(q)}, \mathbf{x}_i^{(q)}, \varphi_{i+1}^{(q)}, \mathbf{x}_{i+1}^{(q)}, \boldsymbol{\mu}_{p_q}, \boldsymbol{\pi}\big) \end{aligned} \tag{8.23}$$

holds, then the mapping pair $(P_q, P_{\hat{q}})$ is skew-symmetric. If a mapping pair is relative to the *local* (or *global*, or *sliding*, or *transport*) mapping, such a mapping pair is termed the *local* (or *global*, or *sliding*, or *transport*) skew-symmetric mapping pair.

Notice that the integer $M_1 = 0, 1, 2, \ldots$ and $\operatorname{mod}(\cdot, \cdot)$ is the modulus function.

THEOREM 8.1. *$2n_1$ mappings P_q $(q = 1, 2, \ldots, 2n_1)$ for the dynamical system in Eq. (8.1) are invariant under the composition of two transformations T_P, i.e., $T_P \circ T_P : P_q \to P_q$.*

PROOF. Under the transformation T_P, mappings P_q and $P_{\hat{q}}$ are skew-symmetric, thus, we have

$$T_P : P_q \to P_{\hat{q}} \quad \text{and} \quad T_P : P_{\hat{q}} \to P_{\hat{\hat{q}}}.$$

From the foregoing relations, for $\varphi_i^{(q)} \in [0, (2M_1 + 1)\pi]$, Definition 8.2 gives for a certain given number $N_2 \in \{0, 1, 2, \ldots\}$

$$\begin{aligned}
&\varphi_i^{(\hat{q})} = \varphi_i^{(q)} + (2N_2 + 1)(2M_1 + 1)\pi \quad \text{and} \quad \mathbf{x}_i^{(\hat{q})} = -\mathbf{x}_i^{(q)},\\
&\mathbf{F}^{(\hat{q})}\big(\varphi_i^{(\hat{q})}, \mathbf{x}_i^{(\hat{q})}, \varphi_{i+1}^{(\hat{q})}, \mathbf{x}_{i+1}^{(\hat{q})}, \boldsymbol{\mu}_{p_q}, \boldsymbol{\pi}\big) = -\mathbf{F}^{(q)}\big(\varphi_i^{(q)}, \mathbf{x}_i^{(q)}, \varphi_{i+1}^{(q)}, \mathbf{x}_{i+1}^{(q)}, \boldsymbol{\mu}_{p_q}, \boldsymbol{\pi}\big),
\end{aligned}$$

and

$$\begin{aligned}
&\varphi_i^{(\hat{\hat{q}})} = \varphi_i^{(\hat{q})} + (2N_2 + 1)(2M_1 + 1)\pi \quad \text{and} \quad \mathbf{x}_i^{(\hat{\hat{q}})} = -\mathbf{x}_i^{(\hat{q})},\\
&\mathbf{F}^{(\hat{\hat{q}})}\big(\varphi_i^{(\hat{\hat{q}})}, \mathbf{x}_i^{(\hat{\hat{q}})}, \varphi_{i+1}^{(\hat{\hat{q}})}, \mathbf{x}_{i+1}^{(\hat{\hat{q}})}, \boldsymbol{\mu}_{p_{\hat{q}}}, \boldsymbol{\pi}\big) = -\mathbf{F}^{(\hat{q})}\big(\varphi_i^{(\hat{q})}, \mathbf{x}_i^{(\hat{q})}, \varphi_{i+1}^{(\hat{q})}, \mathbf{x}_{i+1}^{(\hat{q})}, \boldsymbol{\mu}_{p_{\hat{q}}}, \boldsymbol{\pi}\big).
\end{aligned}$$

Substitution of the first equation into the second one of the foregoing two equations leads to

$$\begin{aligned}
&\varphi_i^{(\hat{\hat{q}})} = \varphi_i^{(q)} + 2(2N_2 + 1)(2M + 1)\pi \quad \text{and} \quad \mathbf{x}_i^{(\hat{\hat{q}})} = \mathbf{x}_i^{(q)},\\
&\mathbf{F}^{(\hat{\hat{q}})}\big(\varphi_i^{(\hat{\hat{q}})}, \mathbf{x}_i^{(\hat{\hat{q}})}, \varphi_i^{(\hat{\hat{q}})}, \mathbf{x}_i^{(\hat{\hat{q}})}, \boldsymbol{\mu}_{p_{\hat{q}}}, \boldsymbol{\pi}\big) = \mathbf{F}^{(q)}\big(\varphi_i^{(q)}, \mathbf{x}_i^{(q)}, \varphi_{i+1}^{(q)}, \mathbf{x}_{i+1}^{(q)}, \boldsymbol{\mu}_{p_q}, \boldsymbol{\pi}\big).
\end{aligned}$$

It is easily proved that $\hat{\hat{q}} = q$ from the property of the modulus function. Since the period for mapping P_q is $(2M_1 + 1)$-period, the phase satisfies the relation $\operatorname{mod}(\varphi_i^{(q)}, 2(2M_1 + 1)\pi) = \varphi_i^{(q)}$. Similarly, the case can be proved for $\varphi_i^{(q)} \in [(2M_1+1)\pi, 2(2M_1+1)\pi]$. Therefore, the foregoing equation indicates that $T_P \circ T_P : P_q \to P_q$ holds. Namely, the mapping P_q $(q = 1, 2, \ldots, 2n_1)$ for dynamical systems in Eq. (8.1) is invariant under the composition of two transformations T_P. □

Consider a post-grazing mapping cluster of a specific local or global mapping P_J $(J \in \{1, 2, \ldots, N\})$. For a local mapping P_{J_1} on the boundary $\partial\Omega_{p_1 p_2}$, the

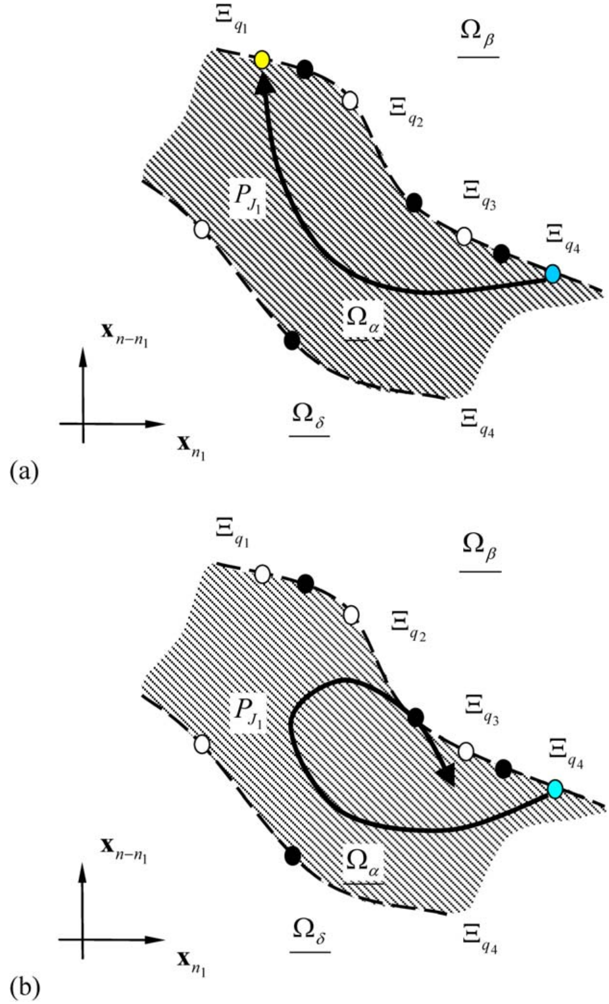


Figure 8.4. Local mapping grazing switching P_{J_1} ($J_1 \in \{1, 2, \ldots, N\}$): (a) pre-grazing mapping, (b) grazing mapping, (c), (d) two possible post-grazing mappings. The black solid circular symbols are singular points. The rest circular points are switching points.

pre-grazing, grazing and post-grazing flows are illustrated in Figs. 8.4(a)–(d). There are many clusters of post-grazing mappings, which determines the property of the post-grazing. Two clusters of the post-grazing mappings for mapping P_{J_1} are sketched. After grazing, the relation between the pre-grazing and post-grazing

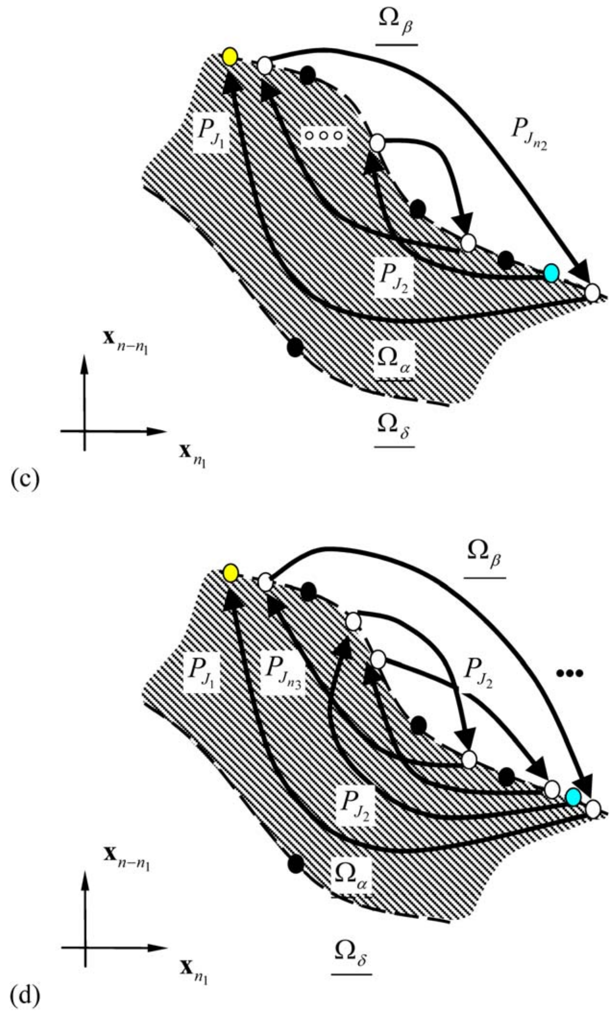


Figure 8.4. (*continued*)

is given by

$$P_{J_1} \xrightleftharpoons[\text{pre-grazing}]{\text{post-grazing}} P_{J_1} \circ \underbrace{P_{J_{n_2}} \circ \cdots \circ P_{J_3} \circ P_{J_2}}_{\text{local mapping cluster}} = P_{J_1(J_{n_2} \cdots J_3 J_2)} \tag{8.24}$$

for $J_1, J_2, \ldots, J_{n_2} \in \{1, 2, \ldots, N\}$. The index J_2 can be J_1 but the index J_i ($J_i = J_3, J_4, \ldots, J_{n_2}$) should not be J_1. $P_{J_2} \neq P_{J_1}$ can be any mappings on

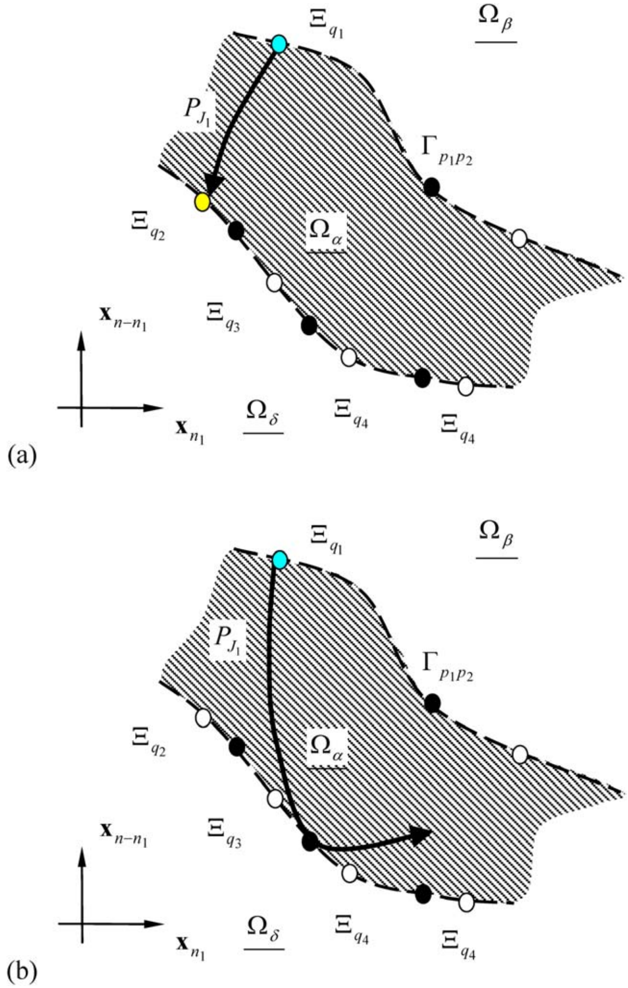
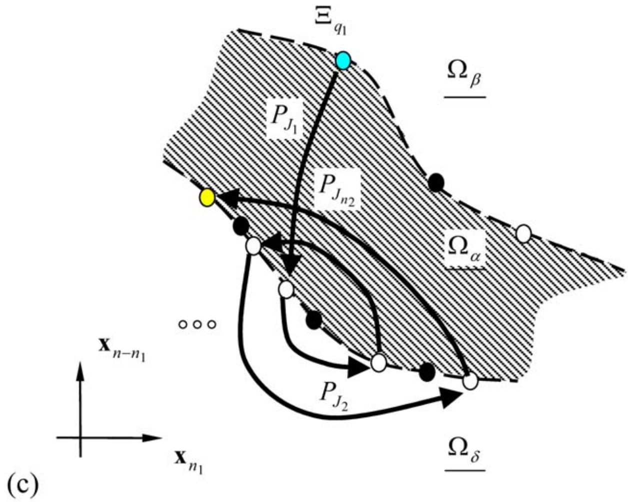


Figure 8.5. Global mapping grazing switching: (a) pre-grazing mapping, (b) grazing mapping, (c), (d) two possible post grazing mappings. The black solid circular symbols are singular points. The rest circular points are switching points.

the *same* boundaries. Similarly, consider a global mapping P_{J_1} to map the flow on the boundary $\partial\Omega_{p_1p_2}$ to another boundary $\partial\Omega_{p_2p_3}$ in domain Ω_{p_2}. The pre-grazing, grazing and post-grazing flows for the global mapping P_{J_1} are illustrated in Figs. 8.5(a)–(d). The relation between the pre-grazing and post-grazing is given



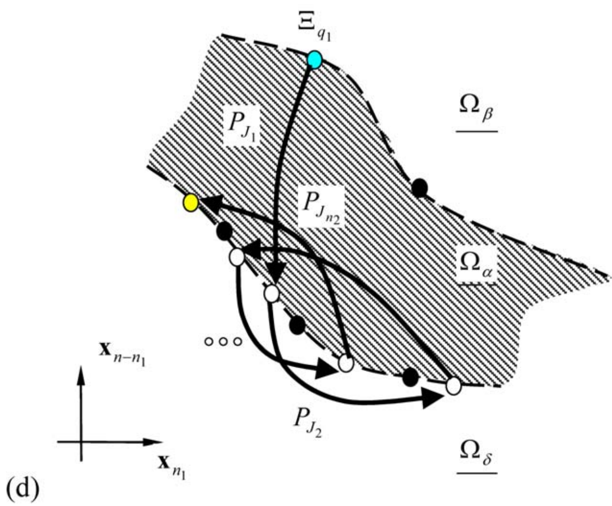


Figure 8.5. (*continued*)

by

$$P_{J_1} \xrightleftharpoons[\text{pre-grazing}]{\text{post-grazing}} \underbrace{P_{J_{n_2}} \circ \cdots \circ P_{J_3}}_{\text{grazing mapping cluster}} \circ P_{J_2} = P_{(J_{n_2} \cdots J_3) J_2} \tag{8.25}$$

for $J_1, J_2, \ldots, J_{n_2} \in \{1, 2, \ldots, N\}$. The index J_2 can be J_1 but the index J_i ($J_i = J_3, J_4, \ldots, J_{n_2}$) cannot be J_1. In post-grazing mapping clusters, the mappings can be local mappings, sliding and transport mapping on the boundary $\partial\Omega_{p_2 p_3}$. Two clusters of post-mapping structure for such a mapping grazing are sketched in Figs. 8.5(c) and (d). For the grazing occurrence of the mapping P_{J_1} on the boundary $\partial\Omega_{p_1 p_2}$, the post-grazing mapping structure is the same as in Eq. (8.24). However, the mapping P_{J_1} is global and the index J_2 cannot be J_1 because P_{J_2} is any mapping rather than P_{J_1} or another global mapping. This mapping can be a local mapping. Without the mapping P_{J_1}, this post-grazing mapping structure is a local mapping grazing structure, which is considered as a case in Eq. (8.24). This concept is extended to the more generalized case. The post-mapping cluster can include any possible mappings rather than the local mappings.

Consider the generic mappings in Fig. 8.3. Once the grazing occurs, the flow of P_J $(J = 1, 2, \ldots, 6)$ switches from an old flow to a new one, and the corresponding post-grazing mapping structures are from Luo (2005f, 2005g)

$$
\begin{aligned}
&P_J \underset{\text{grazing}}{\overset{\text{grazing}}{\rightleftharpoons}} P_J \circ P_{J+1} \circ P_J && \text{for } J = 1, 2, 4, 5,\\
&P_J \underset{\text{grazing}}{\overset{\text{grazing}}{\rightleftharpoons}} P_J \circ P_{\operatorname{mod}(J+1,6)} \circ P_{\operatorname{mod}(J+4,6)} && \text{for } J = 2, 5,\\
&P_J \underset{\text{grazing}}{\overset{\text{grazing}}{\rightleftharpoons}} P_{J-1} \circ P_{J-2} \circ P_J && \text{for } J = 3, 6,\\
&P_J \underset{\text{grazing}}{\overset{\text{grazing}}{\rightleftharpoons}} P_J \circ P_{\operatorname{mod}(J+1,6)} \circ P_{\operatorname{mod}(J+2,6)} && \text{for } J = 3, 6.
\end{aligned}
\tag{8.26}
$$

From the above discussion, the invariance of the post-grazing under the transformation T_P is of great interest. Therefore, we have the following theorem.

THEOREM 8.2. *For symmetric mappings P_q $(q \in \{1, 2, \ldots, 2n_1\}$, $n_1 \leqslant N/2)$ from all N mappings of the dynamical system in Eq.* (8.1)*, if the mapping pair $(P_q, P_{\hat{q}})$ is skew-symmetric with a symmetric transformation T_P, the post-grazing mapping pair is still skew-symmetric with the same transformation.*

PROOF. For the dynamical system in Eq. (8.1), there are n_1 skew-symmetric mapping pairs $(P_q, P_{\hat{q}})$ $(q \in \{1, 2, \ldots, 2n_1\})$ from all the N mappings. Therefore, from Definition 8.2, the skew-symmetry conditions are

$$\varphi_i^{(q_j)} = \varphi_i^{(\hat{q}_j)} + (2M_1 + 1)\pi \quad \text{and} \quad \mathbf{x}_i^{(q_j)} = -\mathbf{x}_i^{(\hat{q}_j)}$$

for $q_j \in \{q_1, q_2, \ldots, q_{n_2}\} \subseteq \{1, 2, \ldots, 2n_1\}$, and

$$
\begin{aligned}
&\mathbf{F}^{(q_j)}\big(\varphi_i^{(q_j)}, \mathbf{x}_i^{(q_j)}, \varphi_{i+1}^{(q_j)}, \mathbf{x}_{i+1}^{(q_j)}, \boldsymbol{\mu}_{p_{q_j}}, \boldsymbol{\pi}\big)\\
&\quad = -\mathbf{F}_i^{(\hat{q}_j)}\big(\varphi_i^{(\hat{q}_j)}, \mathbf{x}_i^{(\hat{q}_j)}, \varphi_{i+1}^{(\hat{q}_j)}, \mathbf{x}_{i+1}^{(\hat{q}_j)}, \boldsymbol{\mu}_{p_{q_j}}, \boldsymbol{\pi}\big)
\end{aligned}
$$

for $q_j \in \{q_1, q_2, \ldots, q_{n_2}\} \subseteq \{1, 2, \ldots, 2n_1\}$.

Consider a post-grazing mapping structure of the skew-symmetry mapping pair $(P_{q_1}, P_{\hat{q}_1})$. For a local mapping $P_{\hat{q}_1}$, the post-grazing mapping structure is

$$P_{\hat{q}_1} \underset{\text{pre-grazing}}{\overset{\text{post-grazing}}{\rightleftharpoons}} P_{\hat{q}_1} \circ P_{\hat{q}_{n_2}} \circ \cdots \circ P_{\hat{q}_3} \circ P_{\hat{q}_2}.$$

The corresponding governing equations are

$$\begin{aligned}
&\mathbf{F}^{(\hat{q}_j)}\big(\varphi_{i+j-2}^{(\hat{q}_j)}, \mathbf{x}_{i+j-2}^{(\hat{q}_j)}, \varphi_{i+j-1}^{(\hat{q}_j)}, \mathbf{x}_{i+j-1}^{(\hat{q}_j)}, \boldsymbol{\mu}_{p_{q_j}}, \boldsymbol{\pi}\big) = 0,\\
&\mathbf{F}^{(\hat{q}_1)}\big(\varphi_{i+n_2-1}^{(\hat{q}_1)}, \mathbf{x}_{i+n_2-1}^{(\hat{q}_1)}, \varphi_{i+n_2}^{(\hat{q}_1)}, \mathbf{x}_{i+n_2}^{(\hat{q}_1)}, \boldsymbol{\mu}_{p_{q_1}}, \boldsymbol{\pi}\big) = 0
\end{aligned}$$

for $j = 2, 3, \ldots, n_2$. With Assumption (A5), the switching points in phase space satisfy

$$\mathbf{x}_{i+j-1}^{(\hat{q}_j)} = \mathbf{x}_{i+j-1}^{(\hat{q}_{j+1})} \quad \text{and} \quad \mathbf{x}_{i+n_2-1}^{(\hat{q}_{n_2})} = \mathbf{x}_{i+n_2-1}^{(\hat{q}_1)}$$

for $j = 2, 3, \ldots, n_2-1$. From Assumption (A1), the system in Eq. (8.1) possesses time-continuity. Therefore the switching phase for $j = 2, 3, \ldots, n_2 - 1$ should be continuous, i.e.,

$$\varphi_{i+j-1}^{(\hat{q}_j)} = \varphi_{i+j-1}^{(\hat{q}_{j+1})} \quad \text{and} \quad \varphi_{i+n_2-1}^{(\hat{q}_{n_2})} = \varphi_{i+n_2-1}^{(\hat{q}_1)}.$$

Multiplication both sides of the governing equations by -1, the switching point state-variable vectors, and then simplification with the skew-symmetry conditions gives

$$\begin{aligned}
&\mathbf{F}^{(q_j)}\big(\varphi_{i+j-2}^{(q_j)}, \mathbf{x}_{i+j-2}^{(q_j)}, \varphi_{i+j-1}^{(q_j)}, \mathbf{x}_{i+j-1}^{(q_j)}, \boldsymbol{\mu}_{p_{q_j}}, \boldsymbol{\pi}\big) = 0,\\
&\mathbf{F}^{(q_1)}\big(\varphi_{i+n_2-1}^{(q_1)}, \mathbf{x}_{i+n_2-1}^{(q_1)}, \varphi_{i+n_2}^{(q_1)}, \mathbf{x}_{i+n_2}^{(q_1)}, \boldsymbol{\mu}_{p_{q_1}}, \boldsymbol{\pi}\big) = 0
\end{aligned}$$

for $j = 2, 3, \ldots, n_2$. The switching points and switching phases satisfy the following for $j = 2, 3, \ldots, n_2$:

$$\begin{aligned}
&\mathbf{x}_{i+j-1}^{(q_j)} = \mathbf{x}_{i+j-1}^{(q_{j+1})} \quad \text{and} \quad \mathbf{x}_{i+n_2-1}^{(q_{n_2})} = \mathbf{x}_{i+n_2-1}^{(q_1)},\\
&\varphi_{i+j-1}^{(q_j)} = \varphi_{i+j-1}^{(q_{j+1})} \quad \text{and} \quad \varphi_{i+n_2-1}^{(q_{n_2})} = \varphi_{i+n_2-1}^{(q_1)}.
\end{aligned}$$

The foregoing equations form the post-grazing mapping structure of mapping P_{q_1}, i.e.,

$$P_{q_1} \underset{\text{pre-grazing}}{\overset{\text{post-grazing}}{\rightleftharpoons}} P_{q_1} \circ P_{q_{n_2}} \circ \cdots \circ P_{q_3} \circ P_{q_2}.$$

Therefore the post-grazing mapping structure $(P_{q_1 q_{n_2} \cdots q_3 q_2}, P_{\hat{q}_1 \hat{q}_{n_2} \cdots \hat{q}_3 \hat{q}_2})$ of the skew-symmetric mapping pair $(P_{q_1}, P_{\hat{q}_1})$ is skew symmetric. Because the local mapping P_{q_1} can be all the local mapping from mapping P_q ($q \in \{1, 2, \ldots, 2n_1\}$), the post-grazing skew-symmetry for the *local* skew-symmetric mapping pair is proved under the symmetric transformation T_P. Similarly, the post-grazing skew-symmetry of the global skew-symmetric mapping pair $(P_{q_1}, P_{\hat{q}_1})$ can be proved with a symmetric transformation T_P. Because the skew-symmetric mapping pair $(P_{q_1}, P_{\hat{q}_1})$ is chosen arbitrarily, the post-grazing mapping structure for a mapping pair $(P_q, P_{\hat{q}})$ $(q \in \{1, 2, \ldots, 2n_1\})$ in the skew-symmetric mapping is also skew-symmetric. □

Since the symmetry invariance of the post-grazing of mapping exists, the combination of the symmetric mapping P_q $(q = 1, 2, \ldots, 2n_1)$ should possess a symmetry invariance under the transformation T_P. For convenience, the following notations for mapping clusters are introduced:

$$\begin{aligned} \hat{P}_{(l_m,n_k)} &= P_{\hat{q}_{n_k}\cdots\hat{q}_{(n_{(k-1)}+1)}(\hat{q}_{n_{k-1}}\cdots\hat{q}_{(n_{(k-2)}+1)})^{l_m}\hat{q}_{n_{(k-2)}}\cdots\hat{q}_{(n_2+1)}(\hat{q}_{n_2}\cdots\hat{q}_2)^{l_1}\hat{q}_1}, \\ P_{(l_m,n_k)} &= P_{q_{n_k}\cdots q_{(n_{(k-1)}+1)}(q_{n_{k-1}}\cdots q_{(n_{(k-2)}+1)})^{l_m}q_{n_{(k-2)}}\cdots q_{(n_2+1)}(q_{n_2}\cdots q_2)^{l_1}q_1}. \end{aligned} \tag{8.27}$$

Therefore, to determine such a symmetrical invariance of the skew-symmetry flow, from the above notation, a theorem is presented as follows:

THEOREM 8.3. *For mappings P_q $(q = 1, 2, \ldots, 2n_1)$ of the dynamical system in Eq. (8.1), if the mapping pair $(P_q, P_{\hat{q}})$ under $(2M_1+1)$-periods is skew-symmetric with a transformation T_P, then the following two mappings,*

$$\Big(\underbrace{\hat{P}_{(l_{m_{2L}},n_{k_{2L}})} \circ P_{(l_{m_{1L}},n_{k_{1L}})} \circ \cdots \circ \hat{P}_{(l_{m_{21}},n_{k_{21}})} \circ P_{(l_{m_{11}},n_{k_{11}})}}_{L\text{-repeating}},$$

$$\underbrace{P_{(l_{m_{1L}},n_{k_{1L}})} \circ \hat{P}_{(l_{m_{2L}},n_{k_{2L}})} \circ \cdots \circ P_{(l_{m_{11}},n_{k_{11}})} \circ \hat{P}_{(l_{m_{21}},n_{k_{21}})}}_{L\text{-repeating}}\Big)$$

are a skew-symmetric mapping pair under the same transformation T_P.

PROOF. Consider a motion relative to a mapping

$$\underbrace{\hat{P}_{(l_{m_{2L}},n_{k_{2L}})} \circ P_{(l_{m_{1L}},n_{k_{1L}})} \circ \cdots \circ \hat{P}_{(l_{m_{21}},n_{k_{21}})} \circ P_{(l_{m_{11}},n_{k_{11}})}}_{L\text{-repeating}}$$

with the initial state $\mathbf{y}_i$ and the final state $\mathbf{y}_{i+\sum_{r=1}^{L}(s_r+\hat{s}_r)}$, and the mapping equation is

$$\mathbf{y}_{i+\sum_{r=1}^{L}s_r+\hat{s}_r} = \underbrace{\hat{P}_{(l_{m_{2L}},n_{k_{2L}})} \circ P_{(l_{m_{1L}},n_{k_{1L}})} \circ \cdots \circ \hat{P}_{(l_{m_{21}},n_{k_{21}})} \circ P_{(l_{m_{11}},n_{k_{11}})}}_{L\text{-repeating}}\mathbf{y}_i.$$

For the mapping clusters $P_{(l_{m_{11}},n_{k_{11}})}$ and $\hat{P}_{(l_{m_{21}},n_{k_{21}})}$ in the rth mapping pair, their mapping numbers are s_r and $\hat{s}_r$ accordingly. Therefore, the governing equations of the foregoing mapping structures for $r \in \{1, 2, \ldots, \sum_{r=1}^{L}(s_r + \hat{s}_r)\}$ are:

$$
\begin{aligned}
&\mathbf{F}^{(q_1)}\big(\varphi^{(q_1)}_{i+\sum_{\sigma=1}^{r-1} s_\sigma+\hat{s}_\sigma}, \mathbf{x}^{(q_1)}_{i+\sum_{\sigma=1}^{r-1} s_\sigma+\hat{s}_\sigma}, \varphi^{(q_1)}_{i+\sum_{\sigma=1}^{r-1} s_\sigma+\hat{s}_\sigma+1}, \mathbf{x}^{(q_1)}_{i+\sum_{\sigma=1}^{r-1} s_\sigma+\hat{s}_\sigma+1},\\
&\quad \boldsymbol{\mu}_{p_{q_1}}, \boldsymbol{\pi}\big) = 0,\\
&\vdots\\
&\mathbf{F}^{(q_{n_{k_r}})}\big(\varphi^{(q_{n_{k_{1r}}})}_{i+\sum_{\sigma=1}^{r-1} s_\sigma+\hat{s}_\sigma+s_r-1}, \mathbf{x}^{(q_{n_{k_{1r}}})}_{i+\sum_{\sigma=1}^{r-1} s_\sigma+\hat{s}_\sigma+s_r-1}, \varphi^{(q_{n_{k_{1r}}})}_{i+\sum_{\sigma=1}^{r-1} s_\sigma+\hat{s}_\sigma+s_r},\\
&\quad \mathbf{x}^{(q_{n_{k_{1r}}})}_{i+\sum_{\sigma=1}^{r-1} s_\sigma+\hat{s}_\sigma+s_r}, \boldsymbol{\mu}_{p_{q_{n_{k_{1r}}}}}, \boldsymbol{\pi}\big) = 0;\\
&\mathbf{F}^{(\hat{q}_1)}\big(\varphi^{(\hat{q}_1)}_{i+\sum_{\sigma=1}^{r-1} s_\sigma+\hat{s}_\sigma+s_r}, \mathbf{x}^{(\hat{q}_1)}_{i+\sum_{\sigma=1}^{r-1} s_\sigma+\hat{s}_\sigma+s_r}, \varphi^{(\hat{q}_1)}_{i+\sum_{\sigma=1}^{r-1} s_\sigma+\hat{s}_\sigma+s_r+1},\\
&\quad \mathbf{x}^{(\hat{q}_1)}_{i+\sum_{\sigma=1}^{r-1} s_\sigma+\hat{s}_\sigma+s_r+1}, \boldsymbol{\mu}_{p_{\hat{q}_1}}, \boldsymbol{\pi}\big) = 0,\\
&\vdots\\
&\mathbf{F}^{(\hat{q}_{n_k})}\big(\varphi^{(\hat{q}_{n_{k_{2r}}})}_{i+\sum_{\sigma=1}^{r} s_\sigma+\hat{s}_\sigma-1}, \mathbf{x}^{(\hat{q}_{n_{k_{2r}}})}_{i+\sum_{\sigma=1}^{r} s_\sigma+\hat{s}_\sigma-1}, \varphi^{(\hat{q}_{n_{k_{2r}}})}_{i+\sum_{\sigma=1}^{r} s_\sigma+\hat{s}_\sigma}, \mathbf{x}^{(\hat{q}_{n_{k_{2r}}})}_{i+\sum_{\sigma=1}^{r} s_\sigma+\hat{s}_\sigma},\\
&\quad \boldsymbol{\mu}_{p_{q_{n_{k_{2r}}}}}, \boldsymbol{\pi}\big) = 0.
\end{aligned}
$$

Application of Eqs. (8.22) and (8.23) to the foregoing equations, and multiplication both sides of the foregoing equations by (-1) leads to

$$
\begin{aligned}
&\mathbf{F}^{(\hat{q}_1)}\big(\varphi^{(\hat{q}_1)}_{i+\sum_{\sigma=1}^{r-1} s_\sigma+\hat{s}_\sigma}, \mathbf{x}^{(\hat{q}_1)}_{i+\sum_{\sigma=1}^{r-1} s_\sigma+\hat{s}_\sigma}, \varphi^{(\hat{q}_1)}_{i+\sum_{\sigma=1}^{r-1} s_\sigma+\hat{s}_\sigma+1}, \mathbf{x}^{(\hat{q}_1)}_{i+\sum_{\sigma=1}^{r-1} s_\sigma+\hat{s}_\sigma+1},\\
&\quad \boldsymbol{\mu}_{p_{q_1}}, \boldsymbol{\pi}\big) = 0,\\
&\vdots\\
&\mathbf{F}^{(\hat{q}_{n_k})}\big(\varphi^{(\hat{q}_{n_{k_{2r}}})}_{i+\sum_{\sigma=1}^{r-1} s_\sigma+\hat{s}_\sigma+\hat{s}_r-1}, \mathbf{x}^{(\hat{q}_{n_{k_{2r}}})}_{i+\sum_{\sigma=1}^{r-1} s_\sigma+\hat{s}_\sigma+\hat{s}_r-1}, \varphi^{(\hat{q}_{n_{k_{2r}}})}_{i+\sum_{\sigma=1}^{r-1} s_\sigma+\hat{s}_\sigma+\hat{s}_r},\\
&\quad \mathbf{x}^{(\hat{q}_{n_{k_{2r}}})}_{i+\sum_{\sigma=1}^{r-1} s_\sigma+\hat{s}_\sigma+\hat{s}_r}, \boldsymbol{\mu}_{p_{q_{n_{k_{2r}}}}}, \boldsymbol{\pi}\big) = 0;\\
&\mathbf{F}^{(q_1)}\big(\varphi^{(q_1)}_{i+\sum_{\sigma=1}^{r-1} s_\sigma+\hat{s}_\sigma+\hat{s}_r}, \mathbf{x}^{(q_1)}_{i+\sum_{\sigma=1}^{r-1} s_\sigma+\hat{s}_\sigma+\hat{s}_r}, \varphi^{(q_1)}_{i+\sum_{\sigma=1}^{r-1} s_\sigma+\hat{s}_\sigma+\hat{s}_r+1},\\
&\quad \mathbf{x}^{(q_1)}_{i+\sum_{\sigma=1}^{r-1} s_\sigma+\hat{s}_\sigma+\hat{s}_r+1}, \boldsymbol{\mu}_{p_{q_1}}, \boldsymbol{\pi}\big) = 0,\\
&\vdots
\end{aligned}
$$

$$\mathbf{F}^{(q_{n_{k_r}})}\big(\varphi^{(q_{n_{k_{1r}}})}_{i+\sum_{\sigma=1}^{r}s_\sigma+\hat{s}_\sigma-1},\mathbf{x}^{(q_{n_{k_{1r}}})}_{i+\sum_{\sigma=1}^{r}s_\sigma+\hat{s}_\sigma-1},\varphi^{(q_{n_{k_{1r}}})}_{i+\sum_{\sigma=1}^{r}s_\sigma+\hat{s}_\sigma},$$
$$\mathbf{x}^{(q_{n_{k_{1r}}})}_{i+\sum_{\sigma=1}^{r}s_\sigma+\hat{s}_\sigma},\boldsymbol{\mu}_{p_{q_{n_{k_{1r}}}}},\boldsymbol{\pi}\big)=0$$

being governing equations for

$$\mathbf{y}_{i+\sum_{r=1}^{L}s_r+\hat{s}_r}=\underbrace{P_{(l_{m_{1L}},n_{k_{1L}})}\circ\hat{P}_{(l_{m_{2L}},n_{k_{2L}})}\circ\cdots\circ P_{(l_{m_{11}},n_{k_{11}})}\circ\hat{P}_{(l_{m_{21}},n_{k_{21}})}}_{L\text{-repeating}}\mathbf{y}_i.$$

Therefore, the following mapping pair,

$$\big(\underbrace{\hat{P}_{(l_{m_{2L}},n_{k_{2L}})}\circ P_{(l_{m_{1L}},n_{k_{1L}})}\circ\cdots\circ\hat{P}_{(l_{m_{21}},n_{k_{21}})}\circ P_{(l_{m_{11}},n_{k_{11}})}}_{L\text{-repeating}},$$
$$\underbrace{P_{(l_{m_{1l}},n_{k_{1L}})}\circ\hat{P}_{(l_{m_{2L}},n_{k_{2L}})}\circ\cdots\circ P_{(l_{m_{11}},n_{k_{11}})}\circ\hat{P}_{(l_{m_{21}},n_{k_{21}})}}_{L\text{-repeating}}\big)$$

is skew-symmetric under a transformation T_P. □

8.4. Steady-state flow symmetry

The symmetry invariance of combined mapping structures has been discussed. The flow symmetry in discontinuous dynamical systems will be presented in this section. Consider a skew-symmetry mapping cluster pair $(P_{(l_m,n_k)},\hat{P}_{(l_m,n_k)})$. The number of mappings is s for two mapping clusters in the mapping pair. The corresponding flows should be of the skew-symmetry. The theorem is presented as follows.

THEOREM 8.4. *For mappings P_q $(q=1,2,\ldots,2n_1)$ of the dynamical system in Eq. (8.1), if the mapping pair $(P_q,P_{\hat{q}})$ under $(2M_1+1)$-periods is skew-symmetric with a transformation T_P, then the asymmetric flows respectively relative to two mappings $P_{(l_m,n_k)}\mathbf{y}=\mathbf{y}$ and $\hat{P}_{(l_m,n_k)}\mathbf{y}=\mathbf{y}$ with the same mapping number s under N_1-periods with a periodicity*

$$\mathbf{y}_{i+s}=\mathbf{y}_i\quad or\quad(\varphi_{i+s},\mathbf{x}_{i+s})^{\mathrm{T}}\equiv(\varphi_i+2N_1\pi,\mathbf{x}_i)^{\mathrm{T}}\tag{8.28}$$

have the corresponding solutions $(\varphi^{\mathrm{I}}_{i+j},\mathbf{x}^{\mathrm{I}}_{i+j})$ and $(\varphi^{\mathrm{II}}_{i+j},\mathbf{x}^{\mathrm{II}}_{i+j})$ for $j\in\{0,1,\ldots,s\}$ on the discontinuous boundaries satisfying the following conditions:

$$\begin{aligned}\hat{\varphi}^{\mathrm{I}}_{i+j}&=\mathrm{mod}\big((2M_1+1)\pi+\hat{\varphi}^{\mathrm{II}}_{i+j},2(2M_1+1)\pi\big),\\ \mathbf{x}^{\mathrm{I}}_{i+j}&=-\mathbf{x}^{\mathrm{II}}_{i+j}.\end{aligned}\tag{8.29}$$

Superscripts I and II denote solutions of $P_{(l_m,n_k)}\mathbf{y}=\mathbf{y}$ and $\hat{P}_{(l_m,n_k)}\mathbf{y}=\mathbf{y}$.

PROOF. From Definition 8.2, the solutions $(\varphi^{\mathrm{I}}_{i+j}, \mathbf{x}^{\mathrm{I}}_{i+j})$ and $(\varphi^{\mathrm{II}}_{i+j}, \mathbf{x}^{\mathrm{II}}_{i+j})$ for the mapping $P_{(l_m,n_k)}\mathbf{y} = \mathbf{y}$ and the skew-symmetry mapping $\hat{P}_{(l_m,n_k)}\mathbf{y} = \mathbf{y}$ must satisfy Eq. (8.29). □

If the local mappings in the mapping cluster exist only on a special discontinuous boundary, the two skew-symmetric, local flows relative to $P_{(l_m,n_k)}\mathbf{y} = \mathbf{y}$ and $\hat{P}_{(l_m,n_k)}\mathbf{y} = \mathbf{y}$ are separated. From the above theorem, once the solutions for one of the two mapping clusters are determined, the solutions for the other mapping clusters can be obtained right away. If the two mapping clusters are combined with global mappings, the symmetric flows relative to a global symmetric mapping structure $\hat{P}_{(l_m,n_k)} \circ P_{(l_m,n_k)}$ are discussed first, and then the asymmetrical flows for such a global symmetric mapping structure are presented. The theorem for the symmetrical flow is stated as follows.

THEOREM 8.5. *For mappings P_q $(q = 1, 2, \ldots, 2n_1)$ of the dynamical system in Eq. (8.1), if the mapping pair $(P_q, P_{\hat{q}})$ under $(2M_1 + 1)$-periods is skew-symmetric under a transformation T_P, then the symmetric flow relative to a mapping $\hat{P}_{(l_m,n_k)} \circ P_{(l_m,n_k)}\mathbf{y} = \mathbf{y}$ with the same mapping number s of two mapping clusters under N_1-periods with a periodicity condition*

$$\mathbf{y}_{i+2s} = \mathbf{y}_i \quad or \quad (\varphi_{i+2s}, \mathbf{x}_{i+2s})^{\mathrm{T}} \equiv (\varphi_i + 2N_1\pi, \mathbf{x}_i)^{\mathrm{T}} \tag{8.30}$$

has the corresponding solutions $(\varphi_{i+j}, \mathbf{x}_{i+j})$ for $j \in \{0, 1, \ldots, 2s\}$ on the discontinuous boundaries satisfying the following conditions:

$$\begin{aligned} \hat{\varphi}_{i+j} &= \operatorname{mod}\big((2M_1+1)\pi + \hat{\varphi}_{i+\operatorname{mod}(s+j,2s)}, 2(2M_1+1)\pi\big), \\ \mathbf{x}_{i+j} &= -\mathbf{x}_{i+\operatorname{mod}(s+j,2s)}. \end{aligned} \tag{8.31}$$

PROOF. For the symmetrical flow of a mapping $\hat{P}_{(l_m,n_k)} \circ P_{(l_m,n_k)}\mathbf{y} = \mathbf{y}$ with Eq. (8.30), the governing equations are

$$\begin{aligned}
&\mathbf{F}^{(q_1)}\big(\varphi_i^{(q_1)}, \mathbf{x}_i^{(q_1)}, \varphi_{i+1}^{(q_1)}, \mathbf{x}_{i+1}^{(q_1)}, \boldsymbol{\mu}_{p_{q_1}}, \boldsymbol{\pi}\big) = 0, \\
&\vdots \\
&\mathbf{F}^{(q_{n_k})}\big(\varphi_{i+s-1}^{(q_{n_k})}, \mathbf{x}_{i+s-1}^{(q_{n_k})}, \varphi_{i+s}^{(q_{n_k})}, \mathbf{x}_{i+s}^{(q_{n_k})}, \boldsymbol{\mu}_{p_{q_{n_k}}}, \boldsymbol{\pi}\big) = 0; \\
&\mathbf{F}^{(\hat{q}_1)}\big(\varphi_{i+s}^{(\hat{q}_1)}, \mathbf{x}_{i+s}^{(\hat{q}_1)}, \varphi_{i+s+1}^{(\hat{q}_1)}, \mathbf{x}_{i+s+1}^{(\hat{q}_1)}, \boldsymbol{\mu}_{p_{q_1}}, \boldsymbol{\pi}\big) = 0, \\
&\vdots \\
&\mathbf{F}^{(\hat{q}_{n_k})}\big(\varphi_{i+2s-1}^{(\hat{q}_{n_k})}, \mathbf{x}_{i+2s-1}^{(\hat{q}_{n_k})}, \varphi_{i+2s}^{(\hat{q}_{n_k})}, \mathbf{x}_{i+2s}^{(\hat{q}_{n_k})}, \boldsymbol{\mu}_{p_{q_{n_k}}}, \boldsymbol{\pi}\big) = 0.
\end{aligned}$$

Substitution of Eq. (8.31) into the foregoing equations and using modulus mod $(s+j, 2s)$ from (8.30) gives for a given number $N_2 \in \{0, 1, 2, \ldots\}$

$$\begin{aligned}
&\mathbf{F}^{(q_1)}\big(\varphi_{i+s}^{(q_1)} + (2N_2+1)(2M_1+1)\pi, -\mathbf{x}_{i+s}^{(q_1)},\\
&\quad \varphi_{i+s+1}^{(q_1)} + (2N_2+1)(2M_1+1)\pi, -\mathbf{x}_{i+s+1}^{(q_1)}, \boldsymbol{\mu}_{p_{q_1}}, \boldsymbol{\pi}\big) = 0,\\
&\vdots\\
&\mathbf{F}^{(q_{n_k})}\big(\varphi_{i+2s-1}^{(q_{n_k})} + (2N_2+1)(2M_1+1)\pi, -\mathbf{x}_{i+2s-1}^{(q_{n_k})},\\
&\quad \varphi_{i+2s}^{(q_{n_k})} + (2N_2+1)(2M_1+1)\pi, -\mathbf{x}_{i+2s}^{(q_{n_k})}, \boldsymbol{\mu}_{p_{q_{n_k}}}, \boldsymbol{\pi}\big) = 0;
\end{aligned}$$

$$\begin{aligned}
&\mathbf{F}^{(\hat{q}_1)}\big(\varphi_{i+2s}^{(\hat{q}_1)} + (2N_2+1)(2M_1+1)\pi, -\mathbf{x}_{i+2s}^{(\hat{q}_1)},\\
&\quad \varphi_{i+2s+1}^{(\hat{q}_1)} + (2N_2+1)(2M_1+1)\pi, -\mathbf{x}_{i+2s+1}^{(\hat{q}_1)}, \boldsymbol{\mu}_{p_{q_1}}, \boldsymbol{\pi}\big) = 0,\\
&\vdots\\
&\mathbf{F}^{(\hat{q}_{n_k})}\big(\varphi_{i+3s-1}^{(\hat{q}_{n_k})} + (2N_2+1)(2M_1+1)\pi, -\mathbf{x}_{i+3s-1}^{(\hat{q}_{n_k})},\\
&\quad \varphi_{i+3s}^{(\hat{q}_{n_k})} + (2N_2+1)(2M_1+1)\pi, -\mathbf{x}_{i+3s}^{(\hat{q}_{n_k})}, \boldsymbol{\mu}_{p_{q_{n_k}}}, \boldsymbol{\pi}\big) = 0.
\end{aligned}$$

Using Eqs. (8.22) and (8.23) in Definition 8.2 into the foregoing equations yields:

$$\begin{aligned}
&\mathbf{F}^{(\hat{q}_1)}\big(\varphi_{i+s}^{(\hat{q}_1)}, \mathbf{x}_{i+s}^{(\hat{q}_1)}, \varphi_{i+s+1}^{(\hat{q}_1)}, \mathbf{x}_{i+s+1}^{(\hat{q}_1)}, \boldsymbol{\mu}_{p_{q_1}}, \boldsymbol{\pi}\big) = 0,\\
&\vdots\\
&\mathbf{F}^{(\hat{q}_{n_k})}\big(\varphi_{i+2s-1}^{(\hat{q}_{n_k})}, \mathbf{x}_{i+2s-1}^{(\hat{q}_{n_k})}, \varphi_{i+2s}^{(\hat{q}_{n_k})}, \mathbf{x}_{i+2s}^{(\hat{q}_{n_k})}, \boldsymbol{\mu}_{p_{q_{n_k}}}, \boldsymbol{\pi}\big) = 0;
\end{aligned}$$

$$\begin{aligned}
&\mathbf{F}^{(q_1)}\big(\varphi_{i}^{(q_1)}, \mathbf{x}_{i}^{(q_1)}, \varphi_{i+1}^{(q_1)}, \mathbf{x}_{i+1}^{(q_1)}, \boldsymbol{\mu}_{p_{q_1}}, \boldsymbol{\pi}\big) = 0,\\
&\vdots\\
&\mathbf{F}^{(q_{n_k})}\big(\varphi_{i+s-1}^{(q_{n_k})}, \mathbf{x}_{i+s-1}^{(q_{n_k})}, \varphi_{i+s}^{(q_{n_k})}, \mathbf{x}_{i+s}^{(q_{n_k})}, \boldsymbol{\mu}_{p_{q_{n_k}}}, \boldsymbol{\pi}\big) = 0.
\end{aligned}$$

After exchanging the order of the above two equations, they are identical to the mapping structure of $\hat{P}_{(l_m,n_k)} \circ P_{(l_m,n_k)}\mathbf{y} = \mathbf{y}$. If $(\varphi_{i+j}, \mathbf{x}_{i+j})^{\mathrm{T}}$ is the solution of periodic flow, then $(\varphi_{i+\bmod(s+j,2s)}, \mathbf{x}_{i+\bmod(s+j,2s)})^{\mathrm{T}}$ is also a skew-symmetry solution satisfying Eq. (8.31). □

The foregoing theorem discussed the symmetrical solutions of period-1 motion associated with the mapping $\hat{P}_{(l_m,n_k)} \circ P_{(l_m,n_k)}$. This structure is quite stable. For instance, one investigated the symmetrical period-1 motion of impacting oscillators (e.g., Luo, 2002; Luo and Chen, 2006; Masri and Caughey, 1966). It was thought this motion may have period-doubling bifurcation. In fact, no period-doubling bifurcation exists (e.g., Masri, 1970; Sentor, 1970). The symmetrical motion will convert into the asymmetrical period-1 motion with the same mapping structures through the saddle-node bifurcation of the first kind and an unstable region. The flow symmetry for such an asymmetric period-2^L ($L = 0, 1, 2, \ldots$) motion is presented in the following theorem.

THEOREM 8.6. *For mappings P_q ($q = 1, 2, \ldots, 2n_1$) of the dynamical system in Eq. (8.1), if the mapping pair $(P_q, P_{\hat{q}})$ under $(2M_1+1)$-periods is skew-symmetric with a transformation T_P, then the two asymmetric flows relative to a mapping*

$$\underbrace{\hat{P}_{(l_m,n_k)} \circ P_{(l_m,n_k)} \circ \cdots \circ \hat{P}_{(l_m,n_k)} \circ P_{(l_m,n_k)}}_{2^L\text{-repeating}} \mathbf{y} = \mathbf{y}$$

for $L = 0, 1, 2, \ldots$ with the same mapping number (i.e., s) in the two mapping clusters under N_1-periods with a periodicity condition

$$\mathbf{y}_{i+2^{L+1}s} = \mathbf{y}_i \quad \mathit{or} \quad (\varphi_{i+2^{L+1}s}, \mathbf{x}_{i+2^{L+1}s})^{\mathrm{T}} \equiv (\varphi_i + 2N_1\pi, \mathbf{x}_i)^{\mathrm{T}} \tag{8.32}$$

possess the following solution properties,

$$\begin{aligned} \hat{\varphi}^{\mathrm{I}}_{i+2(r-1)s+j} &= \operatorname{mod}\big((2M_1+1)\pi + \hat{\varphi}^{\mathrm{II}}_{i+2(r-1)s+\operatorname{mod}(s+j,2s)}, \\ &\qquad\qquad 2(2M_1+1)\pi\big), \\ \mathbf{x}^{\mathrm{I}}_{i+2(r-1)s+j} &= -\mathbf{x}^{\mathrm{II}}_{i+2(r-1)s+\operatorname{mod}(s+j,2s)}; \end{aligned} \tag{8.33}$$

for $r = 1, 2, \ldots, 2^L$ and $j = 0, 1, \ldots, 2s$. Superscripts I and II denote the two asymmetrical solutions.

PROOF. Suppose a mapping relation

$$\underbrace{\hat{P}_{(l_m,n_k)} \circ P_{(l_m,n_k)} \circ \cdots \circ \hat{P}_{(l_m,n_k)} \circ P_{(l_m,n_k)}}_{2^L\text{-repeating}} \mathbf{y} = \mathbf{y}$$

has a set of solutions $(\varphi^{\mathrm{I}}_{i+2(r-1)s+j}, \mathbf{x}^{\mathrm{I}}_{i+2(r-1)s+j})$ for all $j \in \{0, 1, \ldots, 2s\}$ with $r \in \{1, 2, \ldots, 2^N\}$, the governing equations are

$$\begin{aligned} &\mathbf{F}^{(q_1)}\big(\varphi^{\mathrm{I}(q_1)}_{i+2(r-1)s}, \mathbf{x}^{\mathrm{I}(q_1)}_{i+2(r-1)s}, \varphi^{\mathrm{I}(q_1)}_{i+2(r-1)s+1}, \mathbf{x}^{\mathrm{I}(q_1)}_{i+2(r-1)s+1}, \boldsymbol{\mu}_{p_{q_1}}, \boldsymbol{\pi}\big) = 0, \\ &\vdots \\ &\mathbf{F}^{(q_{n_k})}\big(\varphi^{\mathrm{I}(q_{n_k})}_{i+(2r-1)s-1}, \mathbf{x}^{\mathrm{I}(q_{n_k})}_{i+(2r-1)s-1}, \varphi^{\mathrm{I}(q_{n_k})}_{i+(2r-1)s}, \mathbf{x}^{\mathrm{I}(q_{n_k})}_{i+(2r-1)s}, \boldsymbol{\mu}_{p_{q_{n_k}}}, \boldsymbol{\pi}\big) = 0; \end{aligned}$$

$$\mathbf{F}^{(\hat{q}_1)}\big(\varphi^{\mathrm{I}(\hat{q}_1)}_{i+(2r-1)s}, \mathbf{x}^{\mathrm{I}(\hat{q}_1)}_{i+(2r-1)s}, \varphi^{\mathrm{I}(\hat{q}_1)}_{i+(2r-1)s+1}, \mathbf{x}^{\mathrm{I}(\hat{q}_1)}_{i+(2r-1)s+1}, \boldsymbol{\mu}_{p_{q_1}}, \boldsymbol{\pi}\big) = 0,$$

$$\vdots$$

$$\mathbf{F}^{(\hat{q}_{n_k})}\big(\varphi^{\mathrm{I}(\hat{q}_{n_k})}_{i+2rs-1}, \mathbf{x}^{\mathrm{I}(\hat{q}_{n_k})}_{i+2rs-1}, \varphi^{\mathrm{I}(\hat{q}_{n_k})}_{i+2rs}, \mathbf{x}^{\mathrm{I}(\hat{q}_{n_k})}_{i+2rs}, \boldsymbol{\mu}_{p_{q_{n_k}}}, \boldsymbol{\pi}\big) = 0.$$

Substitution of Eq. (8.33) into foregoing equation and using modulus mod$(s + j, 2s)$ from Eq. (8.32) gives for a given number $N_2 \in \{0, 1, 2, \ldots\}$

$$\mathbf{F}^{(q_1)}\big(\varphi^{\mathrm{II}(q_1)}_{i+(2r-1)s} + (2N_2+1)(2M_1+1)\pi, -\mathbf{x}^{\mathrm{II}(q_1)}_{i+(2r-1)s},$$
$$\varphi^{\mathrm{II}(q_1)}_{i+(2r+1)s+1} + (2N_2+1)(2M_1+1)\pi, -\mathbf{x}^{\mathrm{II}(q_1)}_{i+(2r+1)s+1}, \boldsymbol{\mu}_{p_{q_1}}, \boldsymbol{\pi}\big) = 0,$$

$$\vdots$$

$$\mathbf{F}^{(q_{n_k})}\big(\varphi^{\mathrm{II}(q_{n_k})}_{i+2rs-1} + (2N_2+1)(2M_1+1)\pi, -\mathbf{x}^{\mathrm{II}(q_{n_k})}_{i+2rs-1},$$
$$\varphi^{\mathbf{II}(q_{n_k})}_{i+2rs} + (2N_2+1)(2M_1+1)\pi, -\mathbf{x}^{\mathrm{II}(q_{n_k})}_{i+2rs}, \boldsymbol{\mu}_{p_{q_{n_k}}}, \boldsymbol{\pi}\big) = 0;$$

$$\mathbf{F}^{(\hat{q}_1)}\big(\varphi^{\mathrm{II}(\hat{q}_1)}_{i+2rs} + (2N_2+1)(2M_1+1)\pi, -\mathbf{x}^{\mathrm{II}(\hat{q}_1)}_{i+2rs},$$
$$\varphi^{\mathrm{II}(\hat{q}_1)}_{i+2rs+1} + (2N_2+1)(2M_1+1)\pi, -\mathbf{x}^{\mathrm{II}(\hat{q}_1)}_{i+2rs+1}, \boldsymbol{\mu}_{p_{q_1}}, \boldsymbol{\pi}\big) = 0,$$

$$\vdots$$

$$\mathbf{F}^{(\hat{q}_{n_k})}\big(\varphi^{\mathrm{II}(\hat{q}_{n_k})}_{i+(2r+1)s-1} + (2N_2+1)(2M_1+1)\pi, -\mathbf{x}^{\mathrm{II}(\hat{q}_{n_k})}_{i+(2r+1)s-1},$$
$$\varphi^{\mathrm{II}(\hat{q}_{n_k})}_{i+(2r+1)s} + (2N_2+1)(2M_1+1)\pi, -\mathbf{x}^{\mathrm{II}(\hat{q}_{n_k})}_{i+(2r+1)s}, \boldsymbol{\mu}_{p_{q_{n_k}}}, \boldsymbol{\pi}\big) = 0.$$

Using Eqs. (8.22) and (8.23) in Definition 8.2 into the foregoing equations and taking modulus of the index yields

$$\mathbf{F}^{(\hat{q}_1)}\big(\varphi^{\mathrm{II}(\hat{q}_1)}_{i+(2r-1)s}, \mathbf{x}^{\mathrm{II}(\hat{q}_1)}_{i+(2r-1)s}, \varphi^{\mathrm{II}(\hat{q}_1)}_{i+(2r-1)s+1}, \mathbf{x}^{\mathrm{II}(\hat{q}_1)}_{i+(2r-1)s+1}, \boldsymbol{\mu}_{p_{q_1}}, \boldsymbol{\pi}\big) = 0,$$

$$\vdots$$

$$\mathbf{F}^{(\hat{q}_{n_k})}\big(\varphi^{\mathrm{II}(\hat{q}_{n_k})}_{i+2rs-1}, \mathbf{x}^{\mathrm{II}(\hat{q}_{n_k})}_{i+2rs-1}, \varphi^{\mathbf{II}(\hat{q}_{n_k})}_{i+2rs}, \mathbf{x}^{\mathrm{II}(\hat{q}_{n_k})}_{i+2rs}, \boldsymbol{\mu}_{p_{q_{n_k}}}, \boldsymbol{\pi}\big) = 0;$$

$$\mathbf{F}^{(q_1)}\big(\varphi^{\mathrm{II}(q_1)}_{i+2(r-1)s}, \mathbf{x}^{\mathrm{II}(q_1)}_{i+2(r-1)s}, \varphi^{\mathrm{II}(q_1)}_{i+2(r-1)s+1}, \mathbf{x}^{\mathrm{II}(q_1)}_{i+2(r-1)s+1}, \boldsymbol{\mu}_{p_{q_1}}, \boldsymbol{\pi}\big) = 0,$$

$$\vdots$$

$$\mathbf{F}^{(q_{n_k})}\big(\varphi^{\mathrm{II}(q_{n_k})}_{i+(2r-1)s-1}, \mathbf{x}^{\mathrm{II}(q_{n_k})}_{i+(2r-1)s-1}, \varphi^{\mathrm{II}(q_{n_k})}_{i+(2r-1)s}, \mathbf{x}^{\mathrm{II}(q_{n_k})}_{i+(2r-1)s}, \boldsymbol{\mu}_{p_{q_{n_k}}}, \boldsymbol{\pi}\big) = 0.$$

After exchanging the order of the above two sets of equations, it is clear that $(\varphi^{\mathrm{II}}_{i+2(r-1)s+j}, \mathbf{x}^{\mathrm{II}}_{i+2(r-1)s+j})$ for all $j = 0, 1, \ldots, 2s$ with specified $r =$

$1, 2, \ldots, 2^N$ is another solution for mapping structures

$$\underbrace{\hat{P}_{(l_m,n_k)} \circ P_{(l_m,n_k)} \circ \cdots \circ \hat{P}_{(l_m,n_k)} \circ P_{(l_m,n_k)}}_{2^L\text{-repeating}} \mathbf{y} = \mathbf{y}.$$

□

The above theorem discussed about the asymmetrical solutions of periodic and chaotic motions induced by period-doubling bifurcation of the mapping $\hat{P}_{(l_m,n_k)} \circ P_{(l_m,n_k)}\mathbf{y} = \mathbf{y}$. Similarly, the motions pertaining to the asymmetric mapping $\hat{P}_{(l_{m2},n_{k2})} \circ P_{(l_{m1},n_{k1})}\mathbf{y} = \mathbf{y}$ are presented in the following theorem.

THEOREM 8.7. *For mappings P_q $(q = 1, 2, \ldots, 2n_1)$ of the dynamical system in Eq. (8.1), if the mapping pair $(P_q, P_{\hat{q}})$ under $(2M_1+1)$-periods is skew-symmetric with a transformation T_P, then the asymmetric flows respectively relative to two mappings*

$$\underbrace{\hat{P}_{(l_{m2},n_{k2})} \circ P_{(l_{m1},n_{k1})} \circ \cdots \circ \hat{P}_{(l_{m2},n_{k2})} \circ P_{(l_{m1},n_{k1})}}_{2^L\text{-repeating}} \mathbf{y} = \mathbf{y} \quad \textit{and}$$

$$\underbrace{\hat{P}_{(l_{m1},n_{k1})} \circ P_{(l_{m2},n_{k2})} \circ \cdots \circ \hat{P}_{(l_{m1},n_{k1})} \circ P_{(l_{m2},n_{k2})}}_{2^L\text{-repeating}} \mathbf{y} = \mathbf{y}$$

for $L = 0, 1, 2, \ldots$ with different mapping numbers (i.e., s_1 and s_2) in two mapping clusters under N_1-periods with a periodicity condition

$$\begin{aligned} &\mathbf{y}_{i+2^L(s_1+s_2)} = \mathbf{y}_i \quad \textit{or} \\ &(\varphi_{i+2^L(s_1+s_2)}, \mathbf{x}_{i+2^L(s_1+s_2)})^{\mathrm{T}} \equiv (\varphi_i + 2N_1\pi, \mathbf{x}_i)^{\mathrm{T}} \end{aligned} \tag{8.34}$$

are

$$\big(\varphi^{\mathrm{I}}_{i+(r-1)(s_1+s_2)+j}, \mathbf{x}^{\mathrm{I}}_{i+(r-1)(s_1+s_2)+j}\big) \quad \textit{and}$$
$$\big(\varphi^{\mathrm{II}}_{i+(r-1)(s_1+s_2)+j}, \mathbf{x}^{\mathrm{II}}_{i+(r-1)(s_1+s_2)+j}\big)$$

with a solution

$$\begin{aligned} \hat{\varphi}^{\mathrm{I}}_{i+(r-1)(s_1+s_2)+j} &= \mathrm{mod}\big((2M_1+1)\pi + \hat{\varphi}^{\mathrm{II}}_{i+(r-1)(s_1+s_2)+\,\mathrm{mod}\,(s_2+j,s_1+s_2)}, \\ &\qquad 2(2M_1+1)\pi\big), \\ \mathbf{x}^{\mathrm{I}}_{i+(r-1)(s_1+s_2)+j} &= -\mathbf{x}^{\mathrm{II}}_{i+(r-1)(s_1+s_2)+\,\mathrm{mod}\,(s_2+j,s_1+s_2)}; \end{aligned} \tag{8.35}$$

or

$$\begin{aligned} \hat{\varphi}^{\mathrm{II}}_{i+(r-1)(s_1+s_2)+j} &= \mathrm{mod}\big((2M_1+1)\pi + \hat{\varphi}^{\mathrm{I}}_{i+(r-1)(s_1+s_2)+\,\mathrm{mod}\,(s_1+j,s_1+s_2)}, \\ &\qquad 2(2M_1+1)\pi\big), \\ \mathbf{x}^{\mathrm{II}}_{i+(r-1)(s_1+s_2)+j} &= -\mathbf{x}^{\mathrm{I}}_{i+(r-1)(s_1+s_2)+\,\mathrm{mod}\,(s_1+j,s_1+s_2)} \end{aligned} \tag{8.36}$$

for $r = 1, 2, \ldots, 2^L$ *and* $j = 0, 1, \ldots, (s_1 + s_2)$. *Superscripts I and II denote two asymmetrical flows.*

PROOF. Following the same proof procedure of Theorem 8.5, this theorem can be proved. Suppose

$$\underbrace{\hat{P}_{(l_{m2},n_{k2})} \circ P_{(l_{m1},n_{k1})} \circ \cdots \circ \hat{P}_{(l_{m2},n_{k2})} \circ P_{(l_{m1},n_{k1})}}_{2^L\text{-repeating}} \mathbf{y} = \mathbf{y}$$

has a flow solution $(\varphi^{\mathrm{I}}_{i+(r-1)(s_1+s_2)+j}, \mathbf{x}^{\mathrm{I}}_{i+(r-1)(s_1+s_2)+j})$ for all $j \in \{0, 1, \ldots, 2s\}$ with $r \in \{1, 2, \ldots, 2^L\}$, the governing equations are

$$\begin{aligned}
&\mathbf{F}^{(q_1)}\big(\varphi^{\mathrm{I}(q_1)}_{i+(r-1)(s_1+s_2)}, \mathbf{x}^{\mathrm{I}(q_1)}_{i+(r-1)(s_1+s_2)}, \varphi^{\mathrm{I}(q_1)}_{i+(r-1)(s_1+s_2)+1}, \mathbf{x}^{\mathrm{I}(q_1)}_{i+(r-1)(s_1+s_2)+1},\\
&\quad \boldsymbol{\mu}_{p_{q_1}}, \boldsymbol{\pi}\big) = 0,\\
&\vdots\\
&\mathbf{F}^{(q_{n_k})}\big(\varphi^{\mathrm{I}(q_{n_k})}_{i+(r-1)s_2+rs_1-1}, \mathbf{x}^{\mathrm{I}(q_{n_k})}_{i+(r-1)s_2+rs_1-1}, \varphi^{\mathrm{I}(q_{n_k})}_{i+(r-1)s_2+rs_1}, \mathbf{x}^{\mathrm{I}(q_{n_k})}_{i+(r-1)s_2+rs_1},\\
&\quad \boldsymbol{\mu}_{p_{q_{n_k}}}, \boldsymbol{\pi}\big) = 0;\\
&\mathbf{F}^{(\hat{q}_1)}\big(\varphi^{\mathrm{I}(\hat{q}_1)}_{i+(r-1)s_2+rs_1}, \mathbf{x}^{\mathrm{I}(\hat{q}_1)}_{i+(r-1)s_2+rs_1}, \varphi^{\mathrm{I}(\hat{q}_1)}_{i+(r-1)s_2+rs_1+1}, \mathbf{x}^{\mathrm{I}(\hat{q}_1)}_{i+(r-1)s_2+rs_1+1},\\
&\quad \boldsymbol{\mu}_{p_{q_1}}, \boldsymbol{\pi}\big) = 0,\\
&\vdots\\
&\mathbf{F}^{(\hat{q}_{n_k})}\big(\varphi^{\mathrm{I}(\hat{q}_{n_k})}_{i+r(s_1+s_2)-1}, \mathbf{x}^{\mathrm{I}(\hat{q}_{n_k})}_{i+r(s_1+s_2)-1}, \varphi^{\mathrm{I}(\hat{q}_{n_k})}_{i+r(s_1+s_2)}, \mathbf{x}^{\mathrm{I}(\hat{q}_{n_k})}_{i+r(s_1+s_2)}, \boldsymbol{\mu}_{p_{q_{n_k}}}, \boldsymbol{\pi}\big) = 0.
\end{aligned}$$

Substitution of Eq. (8.35) into the foregoing equations and using modulus $\mathrm{mod}(s + j, 2s)$ from Eq. (8.34) gives for a given number $N_2 \in \{0, 1, 2, \ldots\}$

$$\begin{aligned}
&\mathbf{F}^{(q_1)}\big(\varphi^{\mathrm{II}(q_1)}_{i+(r-1)s_1+rs_2} + (2N_2+1)(2M_1+1)\pi, -\mathbf{x}^{\mathrm{II}(q_1)}_{i+(r-1)s_1+rs_2},\\
&\qquad \varphi^{\mathrm{II}(q_1)}_{i+(r-1)s_1+rs_2+1} + (2N_2+1)(2M_1+1)\pi, -\mathbf{x}^{\mathrm{II}(q_1)}_{i+(r-1)s_1+rs_2+1}, \boldsymbol{\mu}_{p_{q_1}}, \boldsymbol{\pi}\big)\\
&\quad = 0,\\
&\vdots\\
&\mathbf{F}^{(q_{n_k})}\big(\varphi^{\mathrm{II}(q_{n_k})}_{i+r(s_1+s_2)-1} + (2N_2+1)(2M_1+1)\pi, -\mathbf{x}^{\mathrm{II}(q_{n_k})}_{i+r(s_1+s_2)-1},\\
&\quad \varphi^{\mathrm{II}(q_{n_k})}_{i+r(s_1+s_2)} + (2N_2+1)(2M_1+1)\pi, -\mathbf{x}^{\mathrm{II}(q_{n_k})}_{i+r(s_1+s_2)}, \boldsymbol{\mu}_{p_{q_{n_k}}}, \boldsymbol{\pi}\big) = 0;\\
&\mathbf{F}^{(\hat{q}_1)}\big(\varphi^{\mathrm{II}(\hat{q}_1)}_{i+r(s_1+s_2)} + (2N_2+1)(2M_1+1)\pi, -\mathbf{x}^{\mathrm{II}(\hat{q}_1)}_{i+r(s_1+s_2)},
\end{aligned}$$

$$\varphi^{\mathrm{II}(\hat{q}_1)}_{i+r(s_1+s_2)+1} + (2N_2+1)(2M_1+1)\pi, -\mathbf{x}^{\mathrm{II}(\hat{q}_1)}_{i+r(s_1+s_2)+1}, \boldsymbol{\mu}_{p_{q_1}}, \boldsymbol{\pi}\big) = 0,$$
$$\vdots$$
$$\mathbf{F}^{(\hat{q}_{n_k})}\big(\varphi^{\mathrm{II}(\hat{q}_{n_k})}_{i+r(s_1+s_2)+s_2-1} + (2N_2+1)(2M_1+1)\pi, -\mathbf{x}^{\mathrm{II}(\hat{q}_{n_k})}_{i+r(s_1+s_2)+s_2-1},$$
$$\varphi^{\mathrm{II}(\hat{q}_{n_k})}_{i+r(s_1+s_2)+s_2} + (2N_2+1)(2M_1+1)\pi, -\mathbf{x}^{\mathrm{II}(\hat{q}_{n_k})}_{i+r(s_1+s_2)+s_2}, \boldsymbol{\mu}_{p_{q_{n_k}}}, \boldsymbol{\pi}\big) = 0.$$

Inserting Eqs. (8.22) and (8.23) of Definition 8.2 into the foregoing equations and taking modulus of the index yields

$$\mathbf{F}^{(\hat{q}_1)}\big(\varphi^{\mathrm{II}(\hat{q}_1)}_{i+rs_2+(r-1)s_1}, \mathbf{x}^{\mathrm{II}(\hat{q}_1)}_{i+rs_2+(r-1)s_1}, \varphi^{\mathrm{II}(\hat{q}_1)}_{i+rs_2+(r-1)s_1+1}, \mathbf{x}^{\mathrm{II}(\hat{q}_1)}_{i+rs_2+(r-1)s_1+1},$$
$$\boldsymbol{\mu}_{p_{q_1}}, \boldsymbol{\pi}\big) = 0,$$
$$\vdots$$
$$\mathbf{F}^{(\hat{q}_{n_k})}\big(\varphi^{\mathrm{II}(\hat{q}_{n_k})}_{i+r(s_1+s_2)-1}, \mathbf{x}^{\mathrm{II}(\hat{q}_{n_k})}_{i+r(s_1+s_2)-1}, \varphi^{\mathrm{II}(\hat{q}_{n_k})}_{i+r(s_1+s_2)}, \mathbf{x}^{\mathrm{II}(\hat{q}_{n_k})}_{i+r(s_1+s_2)}, \boldsymbol{\mu}_{p_{q_{n_k}}}, \boldsymbol{\pi}\big) = 0;$$

$$\mathbf{F}^{(q_1)}\big(\varphi^{\mathrm{II}(q_1)}_{i+(r-1)(s_1+s_2)}, \mathbf{x}^{\mathrm{II}(q_1)}_{i+(r-1)(s_1+s_2)}, \varphi^{\mathrm{II}(q_1)}_{i+(r-1)(s_1+s_2)+1}, \mathbf{x}^{\mathrm{II}(q_1)}_{i+(r-1)(s_1+s_2)+1},$$
$$\boldsymbol{\mu}_{p_{q_1}}, \boldsymbol{\pi}\big) = 0,$$
$$\vdots$$
$$\mathbf{F}^{(q_{n_k})}\big(\varphi^{\mathrm{II}(q_{n_k})}_{i+(r-1)s_1+rs_2-1}, \mathbf{x}^{\mathrm{II}(q_{n_k})}_{i+(r-1)s_1+rs_2-1}, \varphi^{\mathrm{II}(q_{n_k})}_{i+(r-1)s_1+rs_2}, \mathbf{x}^{\mathrm{II}(q_{n_k})}_{i+(r-1)s_1+rs_2},$$
$$\boldsymbol{\mu}_{p_{q_{n_k}}}, \boldsymbol{\pi}\big) = 0.$$

After exchanging the order of the above two sets of equations, it is clear that $(\varphi^{\mathrm{II}}_{i+(r-1)(s_1+s_2)+j}, \mathbf{x}^{\mathrm{II}}_{i+(r-1)(s_1+s_2)+j})$ for all $j \in \{0, 1, \ldots, 2s\}$ with $r \in \{1, 2, \ldots, 2^k\}$ is a set of solutions for the mapping structure

$$\underbrace{\hat{P}_{(l_{m1},n_{k1})} \circ P_{(l_{m2},n_{k2})} \circ \cdots \circ \hat{P}_{(l_{m1},n_{k1})} \circ P_{(l_{m2},n_{k2})}}_{2^N\text{-repeating}} \mathbf{y} = \mathbf{y}.$$

□

The above results can be generalized in the following theorem.

THEOREM 8.8. *For mappings P_q $(q = 1, 2, \ldots, 2n_1)$ of the dynamical system in Eq. (8.1), if the mapping pair $(P_q, P_{\hat{q}})$ under $(2M_1+1)$-periods is skew-symmetric with a transformation T_P, then the asymmetric flows respectively relative to two mappings*

$$\underbrace{\hat{P}_{(l_{m_{2L}},n_{k_{2L}})} \circ P_{(l_{m_{1L}},n_{k_{1L}})} \circ \cdots \circ \hat{P}_{(l_{m_{21}},n_{k_{21}})} \circ P_{(l_{m_{11}},n_{k_{11}})}}_{L\text{-repeating}} \mathbf{y} = \mathbf{y} \quad \textit{and}$$

$$\underbrace{P_{(l_{m_{1L}},n_{k_{1L}})} \circ \hat{P}_{(l_{m_{2N}},n_{kL})} \circ \cdots \circ P_{(l_{m_{11}},n_{k_{11}})} \circ \hat{P}_{(l_{m_{21}},n_{k_{21}})}}_{L\text{-repeating}} \mathbf{y} = \mathbf{y}$$

for $N = 1, 2, \ldots$ *with different mapping numbers* (*i.e.*, s_{1r} *and* s_{2r}) *of the two mapping clusters* ($P_{(l_{m_{1r}},n_{k_{1r}})}$ *and* $\hat{P}_{(l_{m_{2r}},n_{k_{2r}})}$, $r \in \{1, 2, \ldots, N\}$) *under* N_1-*periods with a periodicity condition*

$$\begin{aligned} &\mathbf{y}_{i+\sum_{r=1}^{N} s_{1r}+s_{2r}} = \mathbf{y}_i \quad \textit{or} \\ &(\varphi_{i+\sum_{r=1}^{N} s_{1r}+s_{2r}}, \mathbf{x}_{i+\sum_{r=1}^{N} s_{1r}+s_{2r}})^{\mathrm{T}} \equiv (\varphi_i + 2N_1\pi, \mathbf{x}_i)^{\mathrm{T}} \end{aligned} \tag{8.37}$$

are

$$\begin{aligned} &(\varphi^{\mathrm{I}}_{i+\sum_{\rho=1}^{r-1} s_{1\rho}+s_{2\rho}+j}, \mathbf{x}^{\mathrm{I}}_{i+\sum_{\rho=1}^{r-1} s_{1\rho}+s_{2\rho}+j}) \quad \textit{and} \\ &(\varphi^{\mathrm{II}}_{i+\sum_{\rho=1}^{r-1} s_{1\rho}+s_{2\rho}+j}, \mathbf{x}^{\mathrm{II}}_{i+\sum_{\rho=1}^{r-1} s_{1\rho}+s_{2\rho}+j}) \end{aligned}$$

with a solution structure

$$\begin{aligned} &\hat{\varphi}^{\mathrm{I}}_{i+\sum_{\rho=0}^{r-1} s_{1\rho}+s_{2\rho}+j} \\ &\quad = \mathrm{mod}\big((2M_1+1)\pi + \hat{\varphi}^{\mathrm{II}}_{i+\sum_{\rho=0}^{r-1} s_{1\rho}+s_{2\rho}+j+\ \mathrm{mod}\ (s_{2r}+j, s_{1r}+s_{2r})}, \\ &\qquad 2(2M_1+1)\pi\big), \\ &\mathbf{x}^{\mathrm{I}}_{i+\sum_{\rho=0}^{r-1} s_{1\rho}+s_{2\rho}+j} = -\mathbf{x}^{\mathrm{II}}_{i+\sum_{\rho=0}^{r-1} s_{1\rho}+s_{2\rho}+\ \mathrm{mod}\ (s_{2r}+j, s_{1r}+s_{2r})}; \end{aligned} \tag{8.38}$$

or

$$\begin{aligned} &\hat{\varphi}^{\mathrm{II}}_{i+\sum_{\rho=1}^{r-1} s_{1\rho}+s_{2\rho}+j} \\ &\quad = \mathrm{mod}\big((2M_1+1)\pi + \hat{\varphi}^{\mathrm{I}}_{i+\sum_{\rho=1}^{r-1} s_{1\rho}+s_{2\rho}+\ \mathrm{mod}\ (s_{1r}+j, s_{1r}+s_{2r})}, \\ &\qquad 2(2M_1+1)\pi\big), \\ &\mathbf{x}^{\mathrm{II}}_{i+\sum_{\rho=0}^{r-1}(s_{1\rho}+s_{2\rho})+j} = -\mathbf{x}^{\mathrm{I}}_{i+\sum_{\rho=0}^{r-1} s_{1\rho}+s_{2\rho}+\ \mathrm{mod}\ (s_{1r}+j, s_{1r}+s_{2r})} \end{aligned} \tag{8.39}$$

for $j = 0, 1, \ldots, (s_{1r} + s_{2r})$. *Superscripts I and II denote two skew-symmetry solutions of flows.*

PROOF. From Theorem 8.3, the two mapping structures are skew-symmetric. Therefore, the two solutions of the two mapping structures are skew-symmetric, which implies that Eqs. (8.37)–(8.39) hold. The alternative proof can be completed through the procedure used in proof of Theorem 8.7. □

The symmetry of flow in nonsmooth dynamical systems with mapping clusters on many discontinuous boundaries is discussed. The grazing does not change the symmetry invariance of mapping structures in such dynamical systems, and the periodic and chaotic motions in such a dynamical system possess the symmetry invariance as same as the basic mappings. Based on this investigation, the group structure of mapping combination exists. Thus the further investigation on such an issue should be carried out. The illustrations for the symmetry of periodic motions in symmetric, nonlinear dynamic systems can be found in Chapter 7. The detailed presentations can be also seen in Luo (2005e).

8.5. Strange attractor fragmentation

The symmetry of the post-grazing flow has been presented in the previous section. In this section, the fragmentation of strange attractors of chaotic motions in discontinuous dynamic systems will be discussed. Before the discussion of the fragmentation mechanism, the initial and final set of grazing mapping will be introduced first. Because the grazing is strongly dependent on the singular sets in Eqs. (8.6)–(8.8), the definitions are given as follows.

DEFINITION 8.3. For a discontinuous dynamical system in Eq. (8.1) (or Eq. (2.1)), if $\Xi_{q_1} \subseteq \partial\Omega_{p_2 p_1}$ and $\Xi_{q_2} \subseteq \partial\Omega_{p_1\alpha}$ ($\alpha \in \{p_2, p_3\}$ and $p_i \in \{1, 2, \ldots, m\}$ for $i \in \{1, 2, 3\}$), the following subset ${}^{(\mathrm{i})}\Gamma_J$ for mapping $P_J : \Xi_{q_1} \to \Xi_{q_2}$ ($J \in \{1, 2, \ldots, N\}$ and $q_1, q_2 \in \{0, 1, 2, \ldots, M\}$) is called *the initial set of grazing mapping*,

$$
\begin{aligned}
{}^{(\mathrm{i})}\Gamma_J \equiv \Big\{ \big(\mathrm{mod}(\varphi_i, 2\pi), \mathbf{x}_i\big) \;\Big|\; & P_J \mathbf{y}_i = \mathbf{y}_{i+1}, \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{p_1\alpha}} \cdot \dot{\mathbf{x}}_{i+1}(t_{(i+1)\pm}) = 0 \\
& \text{and } \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{p_1\alpha}} \cdot \ddot{\mathbf{x}}_{i+1} \neq 0 \Big\} \subset \Xi_{q_1}
\end{aligned} \tag{8.40}
$$

where $\mathbf{y}_i = (\varphi_i, \mathbf{x}_i)^{\mathrm{T}}$. The corresponding grazing set is defined as

$$
\begin{aligned}
{}^{(\mathrm{f})}\mathrm{G}_J \equiv \Big\{ \big(\mathrm{mod}(\varphi_{i+1}, 2\pi), \mathbf{x}_{i+1}\big) \;\Big|\; & P_J \mathbf{y}_i = \mathbf{y}_{i+1}, \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{p_1\alpha}} \cdot \dot{\mathbf{x}}_{i+1}(t_{(i+1)\pm}) = 0 \\
& \text{and } \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{p_1\alpha}} \cdot \ddot{\mathbf{x}}_{i+1}(t_{(i+1)\pm}) \neq 0 \Big\} \subset \Xi_{q_2}.
\end{aligned} \tag{8.41}
$$

DEFINITION 8.4. For a discontinuous dynamical system in Eq. (8.1) (or Eq. (2.1)), if $\Xi_{q_1} \subseteq \partial\Omega_{p_2 p_1}$ and $\Xi_{q_2} \subseteq \partial\Omega_{p_1\alpha}$ (($\alpha \in \{p_2, p_3\}$ and $p_i \in \{1, 2, \ldots, m\}$ for $i \in \{1, 2, 3\}$), the following subset ${}^{(\mathrm{f})}\Gamma_J$ for mapping $P_J : \Xi_{q_1} \to \Xi_{q_2}$ ($J \in \{1, 2, \ldots, N\}$ and $q_1, q_2 \in \{0, 1, 2, \ldots, M\}$) is called the final set of grazing post-mapping:

$$
\begin{aligned}
{}^{(\mathrm{f})}\Gamma_J \equiv \Big\{ \big(\mathrm{mod}(\varphi_{i+1}, 2\pi), \mathbf{x}_{i+1}\big) \;\Big|\; & P_J \mathbf{y}_i = \mathbf{y}_{i+1}, \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{p_1 p_2}} \cdot \dot{\mathbf{x}}_i(t_{i\pm}) = 0 \\
& \text{and } \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{p_1 p_2}} \cdot \ddot{\mathbf{x}}_I(t_{i\pm}) \neq 0 \Big\} \subset \Xi_{q_2}
\end{aligned} \tag{8.42}
$$

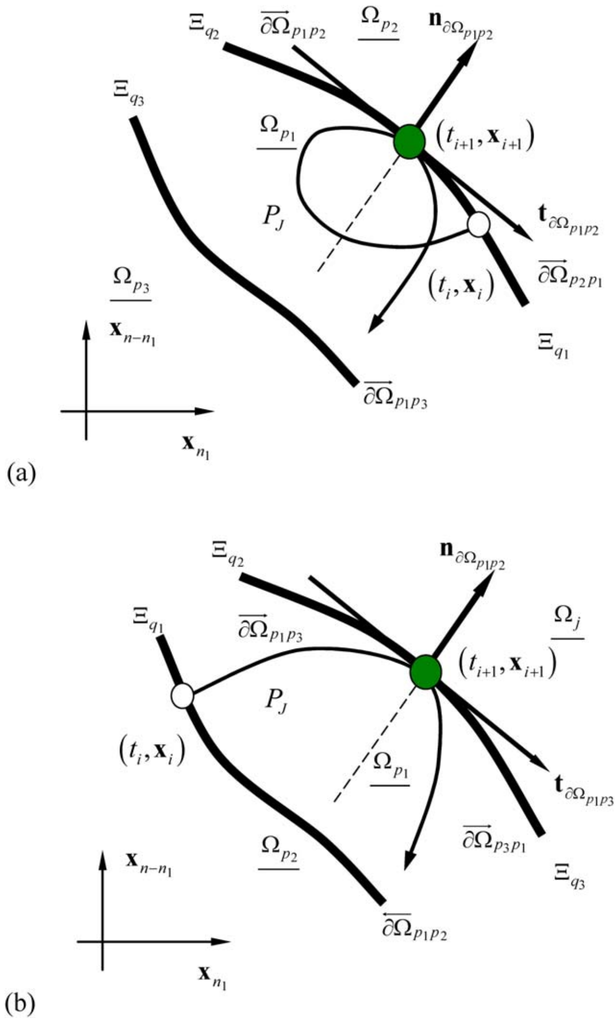


Figure 8.6. (a) Local and (b) global grazing mappings. The filled solid circular symbols are grazing points. The hollow circular symbols are initial switching points.

where $\mathbf{y}_i = (\Omega t_i, \mathbf{x}_i)^{\mathrm{T}}$. The corresponding grazing set of the post-grazing is defined as

$$
\begin{aligned}
{}^{(i)}\mathrm{G}_J \equiv \Big\{ \big(\mathrm{mod}(\Omega t_i, 2\pi), \mathbf{x}_i\big) \;\Big|\; & P_J \mathbf{y}_i = \mathbf{y}_{i+1}, \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{p_1p_2}} \cdot \dot{\mathbf{x}}_i(t_{i\pm}) = 0 \\
& \text{and } \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{p_1p_2}} \cdot \ddot{\mathbf{x}}_i(t_{i\pm}) \neq 0 \Big\} \subset \Xi_{q_1}. \qquad (8.43)
\end{aligned}
$$

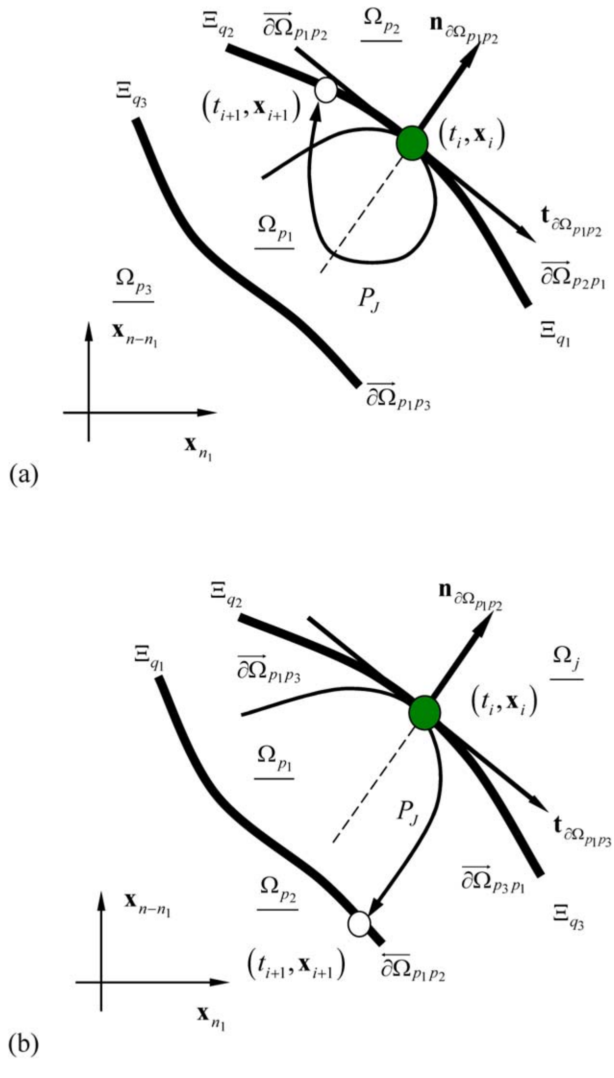


Figure 8.7. (a) Local and (b) global grazing post-mappings. The filled solid circular symbols are grazing points. The hollow circular symbols are initial switching points.

For global and local grazing mappings, the grazing and post-grazing mapping are sketched in Figs. 8.6 and 8.7 through mapping P_J ($J \in \{J_1, \ldots, J_m\}$) on the boundary $\partial\Omega_{p_2p_1}$. The grazing in the domain Ω_{p_1} occurs at the final points of the grazing mapping P_J on the boundary $\partial\Omega_{p_1\alpha}$ ($\alpha \in p_1, p_3$). The above definitions for both the initial grazing sets of grazing mapping and the final sets of grazing post-mapping are illustrated. The hollow symbols are the initial point for grazing mapping or the final point of grazing post-mapping. The circular symbols are the grazing point of the grazing and post grazing mappings. The governing equations of mapping P_J with the final point on the boundary $\partial\Omega_{p_1\alpha}$ in Ω_α ($\alpha \in \{p_2, p_3\}$) is expressed by

$$\begin{aligned} &\mathbf{F}^{(J)}(\varphi_i, \mathbf{x}_i, \varphi_{i+1}, \mathbf{x}_{i+1}) = 0, \quad \mathbf{F}^{(J)} \in \Re^n, \\ &\varphi_{p_1p_2}(\mathbf{x}_i) = 0, \qquad \varphi_{p_1\alpha}(\mathbf{x}_{i+1}) = 0. \end{aligned} \tag{8.44}$$

From Theorem 2.11, the grazing necessary conditions for mapping P_J at the singular set $\Gamma^{(0)}_{p_1\alpha}$ in the subdomain Ω_α ($\alpha, \beta \notin \{p_1, p_2\}, \alpha \neq \beta$) are:

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{p_1p_2}} \cdot \mathbf{F}^{(\alpha)}(\varphi_{i+1}) = 0. \tag{8.45}$$

To guarantee the occurrence of the grazing flow at the singular points, the sufficient conditions should be considered as follows:

$$\begin{aligned} &\text{either} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{p_1p_2}} \cdot \left[\mathrm{D}\mathbf{F}^{(\alpha)}(\varphi_{i+1}) - \mathrm{D}\mathbf{F}^{(0)}(\varphi_{i+1})\right] < 0 \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta \\ &\text{or} \qquad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{p_1p_2}} \cdot \left[\mathrm{D}\mathbf{F}^{(\alpha)}(\varphi_{i+1}) - \mathrm{D}\mathbf{F}^{(0)}(\varphi_{i+1})\right] > 0 \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha. \end{aligned} \tag{8.46}$$

From Eqs. (8.44) and (8.45), the initial set of grazing mapping is on an $(n-1)$-surface because of the $(n+3)$ equations with $2(n+1)$ unknowns. Equation (8.46) is the sufficient condition for the initial set. This $(n-1)$-surface in phase space $(\mathrm{mod}(\varphi_i, 2\pi), \mathbf{x}_i \cap \partial\Omega_{p_1p_2})$ is called the *initial grazing manifold* which will be used in discussion of the strange attractor fragmentation. Owing to $\varphi_i = \Omega t_i$, in computation, the switching time conditions $t_{i+1} > t_i$ should be inserted. The boundary $\partial\Omega_{p_1\alpha}$ is determined by $\varphi_{p_1\alpha}(\mathbf{x}_i) = 0$. Similarly, when φ_i and φ_{i+1} in Eqs. (8.44)–(8.46) are exchanged, *the final grazing manifold* can be determined through the final set of grazing post-mapping.

To investigate the fragmentation of the strange attractors of chaos through the switching sets, consider each switching set Ξ_q consisting of a set of finite, independent compact subsets, (i.e., $\Xi_q = \bigcup_{k=1}^{K} S_q^{(k)}$ and $K < \infty$) and the subsets have the following properties:

$$S_q^{(k)} \subset \Xi_q, \quad S_q^k \cap S_q^{(l)} = \varnothing, \quad k \neq l \in \{1, 2, \ldots, K\}. \tag{8.47}$$

For periodic motions, the fixed points in the compact subset $S_{ij}^{(k)}$ for each specific k are countable and finite, and both the measure and Hausdorff dimension of all the subsets are zero. However, from the definition of strange attractors for chaotic motions, the fixed points in such compact subsets are infinite and uncountable, and the corresponding Hausdorff dimension is nonzero. In addition, the compact subsets are not hyperbolic. Based on the mapping P_J on the compact subsets of the strange attractor, the corresponding flows in the domains are dense. All the subsets form a strange attractor of the chaotic motion. The denseness of the flows on subsets in the strange attractor will cause more possibilities for the strange attractor to access at least one of the initial grazing manifolds. However, the flows in periodic motions have less possibility to access the initial grazing manifolds. Without chaotic motions, the transition between the pre- and post-grazing periodic motions can be carried out through a grazing jump. This issue of periodic motion grazing will be further discussed. To describe the fragmentation of the strange attractors in chaotic motions of nonsmooth dynamical system, the compact subsets of the strange attractors are used as the initial and final sets of the local and global mappings, i.e.,

DEFINITION 8.5. For a discontinuous dynamical system in Eq. (8.1), the subsets $S_{q_1}^{(k)} \subset \Xi_{q_1}$ $(k = 1, 2, \ldots, K)$ and $S_{q_2}^{(l)} \subset \Xi_{q_2}$ $(l = 1, 2, \ldots, L)$ are termed the *initial* and *final* subsets of the mapping P_J $(J \in \{1, 2, \ldots, N\})$ if there is a mapping $P_J : S_{q_1}^{(k)} \to S_{q_2}^{(l)}$ $(q_1, q_2 \in \{0, 1, 2, \ldots, M\}, q_1 \neq q_2)$.

For convenience, after grazing, the post-grazing mapping structure is defined as

$$^{\mathrm{G}}P_J \equiv P_{J_{n+1}} \circ \underbrace{P_{J_n} \circ \cdots \circ P_{J_2} \circ P_{J_1}}_{\text{post-grazing mapping cluster}} = P_{J_{n+1}} \circ {}^{C}P_{(J_n \cdots J_1)}. \tag{8.48}$$

Once the intersection exists between the invariant subsets of the strange attractor and one of the *initial* grazing manifolds of the generic mappings, the fragmentation of the strange attractor will occur. For a subset $S_{q_1}^{(k)}$ and an initial, grazing manifold ${}^{(\mathrm{i})}\Gamma_J$ relative to a mapping P_J, if $S_{q_1}^{(k)} \cap {}^{(\mathrm{i})}\Gamma_J \neq \varnothing$, then the final subset of the mapping P_J will be fragmentized. If one of the all initial, grazing manifolds is tangential to one of the strange attractor subsets, the strange attractor fragmentation may appear or disappear. Therefore, a mathematical definition of strange attractor fragmentation is given as follows.

DEFINITION 8.6. For a discontinuous dynamical system in Eq. (8.1), there is mapping cluster $P_{(l_m, n_k)}$ with s mappings to generate the strange attractor of chaotic motions on the switching set Ξ_q $(q \in \{0, 1, 2, \ldots, M\})$. For a mapping $P_J : S_{q_1}^{(k)} \to S_{q_2}^{(l)}$ $(J \in \{1, 2, \ldots, N\})$, if ${}^{(\mathrm{i})}\Pi_{q_1}^{(k)} \equiv S_{q_1}^{(k)} \cap {}^{(\mathrm{i})}\Gamma_J \neq \varnothing$ and $S_{q_1}^{(k)} = {}^{\mathrm{F}}S_{q_1}^{(k)} \cup {}^{\mathrm{U}}S_{q_1}^{(k)} \cup {}^{(\mathrm{i})}\Pi_{q_1}^{(k)}$, then ${}^{(\mathrm{f})}\Pi_{q_2}^{(l)} \equiv S_{q_2}^{(l)} \cap {}^{(\mathrm{f})}\Gamma_J \neq \varnothing$ with $S_{q_2}^{(l)} = {}^{\mathrm{F}}S_{q_2}^{(l)} \cup {}^{\mathrm{U}}S_{q_2}^{(l)} \cup {}^{(\mathrm{f})}\Pi_{q_2}^{(l)}$ exist to make the following mappings hold:

$$P_J : {}^{(\mathrm{i})}\Pi_{q_1}^{(k)} \to {}^{(\mathrm{f})}G_J^{(k)} \quad \text{for } {}^{(\mathrm{f})}G_J^{(k)} \subset {}^{(\mathrm{f})}G_J, \tag{8.49}$$

$$P_J : {}^{\mathrm{U}}S_{q_1}^{(k)} \to {}^{\mathrm{U}}S_{q_2}^{(l)} \quad \text{and} \quad {}^{G}P_J : {}^{\mathrm{F}}S_{q_1}^{(k)} \to {}^{\mathrm{F}}S_{q_2}^{(l)}. \tag{8.50}$$

For the post-grazing mapping cluster ${}^{G}P_J = P_{J_{n+1}} \circ P_{J_n} \circ \cdots \circ P_{J_2} \circ P_{J_1}$, if there is a mapping chain as

$${}^{\mathrm{F}}S_{q_1}^{(k)} \xrightarrow{P_{J_1}} {}^{\mathrm{F}}\mathfrak{A}_{\rho_1}^{(k_1)} \ldots {}^{\mathrm{F}}\mathfrak{A}_{\rho_{n-1}}^{(k_{n-1})} \xrightarrow{P_{J_n}} {}^{\mathrm{F}}\mathfrak{A}_{\rho_n}^{(k_n)} \xrightarrow{P_{J_{n+1}}} {}^{\mathrm{F}}S_{q_2}^{(l)}, \tag{8.51}$$

the union of all the switching sets generated by the mapping cluster ${}^{G}P_J$, $\mathfrak{A}_\rho = \bigcup_{i=1}^{n} \mathfrak{A}_{\rho_i}^{(k_i)}$ ($\mathfrak{A}_{\rho_i}^{(k_i)} \subset \Xi_{\rho_i}$ for $\rho_i \in \{0, 1, 2, \ldots, M\}$), is termed a *fragmentation set* of the invariant set $S_{q_1}^{(k)} \subset \Xi_{q_1}$ under the mapping cluster $P_{(l_m,n_k)}$.

To demonstrate the concept introduced above, consider the invariant subsets of a strange attractor on the boundary $\partial\Omega_{p_1p_2}$ for the following mapping structure:

$$P_{\circ\cdots\circ J_4(J_1J_2)^m J_1J_3\circ\cdots\circ} = \circ\cdots\circ P_{J_4} \circ \underbrace{(P_{J_1} \circ P_{J_2})}_{m \text{ sets}} \circ P_{J_1} \circ P_{J_3} \circ \cdots \circ. \tag{8.52}$$

The invariant sets are generated by

$$P^{(l)}_{\circ\cdots\circ J_4(J_1J_2)^m J_1J_3\circ\cdots\circ} = \underbrace{P_{\circ\cdots\circ J_4(J_1J_2)^m J_1J_3\circ\cdots\circ} \circ \cdots \circ P_{\circ\cdots\circ J_4(J_1J_2)^m J_1J_3\circ\cdots\circ}}_{l\to\infty} \tag{8.53}$$

and the corresponding invariant sets on the boundary $\partial\Omega_{p_1p_2}$ are

$$\Xi_{q_1} = \bigcup_{k=1}^{m+1} S_{q_1}^{(k)} \quad \text{and} \quad \Xi_{q_2} = \bigcup_{k=1}^{m+1} S_{q_2}^{(k)}. \tag{8.54}$$

The foregoing invariant subsets are illustrated in Fig. 8.8(a) through the filled areas on the switching plane $(\mathrm{mod}(\varphi_i, 2\pi), \mathbf{x}_i) \cap \partial\Omega_{p_1p_2}$. The mapping relation for the mapping structure in Eq. (8.52) is also presented. The initial grazing manifolds ${}^{(\mathrm{i})}\Gamma_J$ ($J \in \{J_1, J_2, J_3, J_4\}$) are sketched by the dashed lines in Fig. 8.8(b) in the corresponding switching planes. For this case, $\Xi_{q_1} \cap {}^{(\mathrm{i})}\Gamma_J = \varnothing$ and $\Xi_{q_2} \cap {}^{(\mathrm{i})}\Gamma_J = \varnothing$, no fragmentation of the strange attractor occurs for the aforementioned mapping structure. If $\Xi_{q_1} \cap {}^{(\mathrm{i})}\Gamma_J \neq \varnothing$ and/or $\Xi_{q_2} \cap {}^{(\mathrm{i})}\Gamma_J \neq \varnothing$, the fragmentation will occur at $\partial\Omega_{p_1p_2}$. The fragmentation of strange attractors on $\partial\Omega_{p_1p_2}$ is illustrated in Fig. 8.9. Suppose ${}^{(\mathrm{i})}\Pi_{q_2}^{(K)} = S_{q_2}^{(K)} \cap {}^{(\mathrm{i})}\Gamma_{J_2} \neq \varnothing$ be represented by the solid symbols, $S_{q_2}^{(k)} = {}^{\mathrm{F}}S_{q_2}^{(k)} \cup {}^{\mathrm{U}}S_{q_2}^{(k)} \cup {}^{(\mathrm{i})}\Pi_{J_2}^{(k)}$. The initial grazing manifold is depicted by the dashed curve. Furthermore, after fragmentation, two new invariant sets $\mathfrak{A}_{q_2}^{(k_1)} \subset \Xi_{q_2}$ and $\mathfrak{A}_{q_1}^{(k_2)} \subset \Xi_{q_1}$ exist, which are showed by the shaded areas. The nonfragmentized mapping are: $P_{J_1} : {}^{\mathrm{U}}S_{q_2}^{(k)} \to {}^{\mathrm{U}}S_{q_1}^{(k+1)}$ and the mappings relative to the fragmentation are: $P_{J_2} : {}^{\mathrm{F}}S_{q_2}^{(k)} \to \mathfrak{A}_{q_1}^{(k_1)}$, $P_{J_1} : \mathfrak{A}_{q_1}^{(k_1)} \to \mathfrak{A}_{q_2}^{(k_2)}$

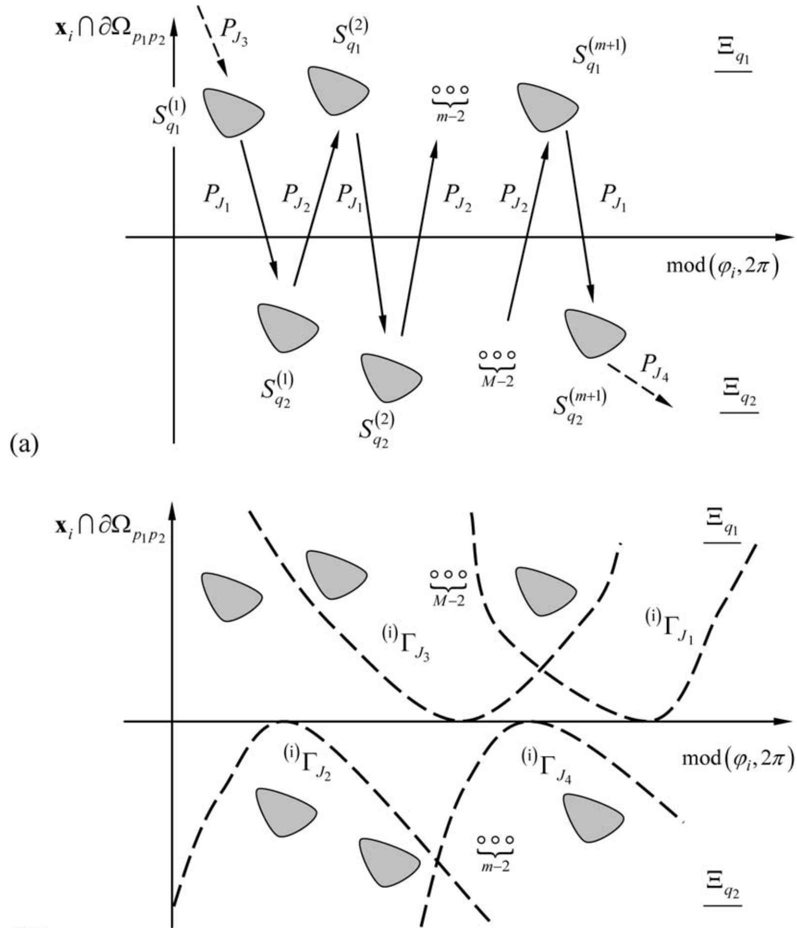


Figure 8.8. (a) Invariant subsets, and (b) invariant initial manifolds of grazing mapping on the boundary $\partial\Omega_{p_1p_2}$.

and $P_{J_2} : \mathfrak{A}_{q_2}^{(k_1)} \to {}^{\mathrm{F}}S_{q_1}^{(k+1)}$. From the new subsets to the nonfragmentized subsets, the final manifold of the post-grazing ${}^{(\mathrm{f})}\Gamma_{J_2}$, expressed by the dotted dash curve, separates the invariant subset into two parts ${}^{\mathrm{U}}S_{q_1}^{(k+1)} \cup {}^{\mathrm{F}}S_{q_1}^{(k+1)}$ plus the intersected set ${}^{(\mathrm{f})}\Pi_{q_1}^{(k+1)} = S_{q_1}^{(k+1)} \cap {}^{(\mathrm{f})}\Gamma_{J_1} \neq \varnothing$. Note that if ${}^{(\mathrm{i})}\Pi_{q_2}^{(k)}$ possesses n values, there are n pieces of nonintersected, invariant subsets in $\mathfrak{A}_{q_2}^{(1)}$. As $n \to \infty$, countable infinity pieces of nonintersected invariant subsets are obtained by such an attractor fragmentation. The initial sets of grazing mapping are presented and

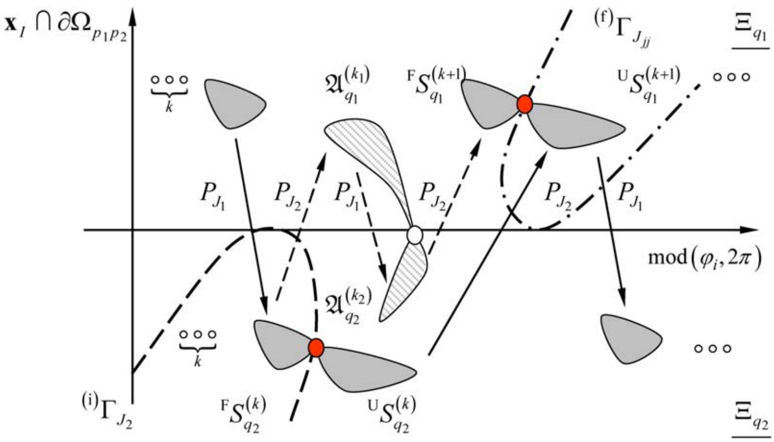


Figure 8.9. Invariant sets fragmentation on the boundary $\partial\Omega_{p_1 p_2}$.

the corresponding, initial grazing manifolds are presented. Finally, the grazing-induced fragmentation of strange attractors of chaotic motions in discontinuous dynamical systems is discussed. The mathematical theory for such a fragmentation of strange attractors should be further developed.

8.6. Fragmentized strange attractors

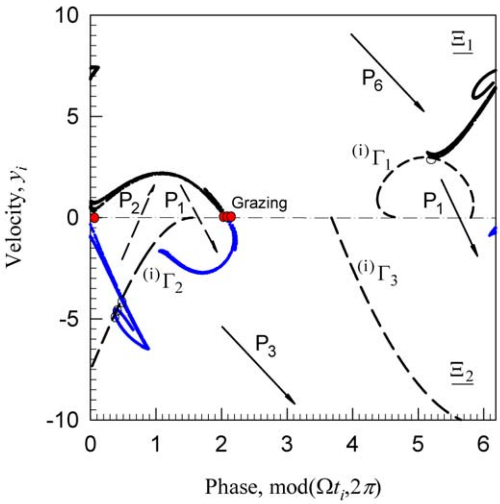
To demonstrate a fragmentized strange attractor, consider a periodically excited, piecewise linear system in Eqs. (2.45) and (2.46) again, i.e.,

$$\ddot{x} + 2d\dot{x} + k(x) = a\cos\Omega t, \tag{8.55}$$

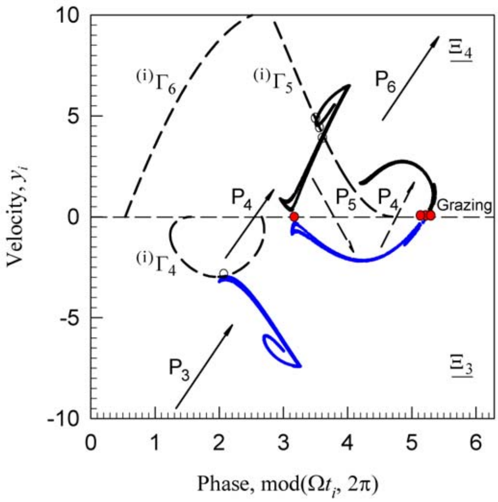
where $\dot{x} = \mathrm{d}x/\mathrm{d}t$. The parameters ($\Omega$ and a) are excitation frequency and amplitude, respectively. The restoring force is

$$k(x) = \begin{cases} cx - e, & \text{for } x \in [E, \infty), \\ 0, & \text{for } x \in [-E, E], \\ cx + e, & \text{for } x \in (-\infty, -E]. \end{cases} \tag{8.56}$$

For illustration of the fragmentation of strange attractors in chaotic motion, the Poincaré mapping sections for four subswitching planes are plotted. Herein, numerical simulations are based on the closed-form solution for switching planes. As in Fig. 8.3, the subsets (i.e., Ξ_2 and Ξ_1) of the switching plane Ξ_{12} for strange attractors are on the upper and lower dashed lines. Similarly, the subsets (i.e., Ξ_3 and Ξ_4) of the switching plane Ξ_{34} separated by the dashed line are presented as well. The dashed curves are the *initial* grazing, switching manifolds



(a)



(b)

Figure 8.10. Chaotic motion associated with mappings $P_{6(45)431}$, P_{6431} and $P_{643(12)1}$: (a) subsets of switching plane Ξ_{12} (Ξ_1 and Ξ_2), and (b) subsets of switching plane Ξ_{34} (Ξ_3 and Ξ_4). ($a = 20$, $c = 100$, $E = 1$, $d = 0.5$, $\Omega t_i \approx 6.1117$, $x_i = 1$, $y_i \approx 6.5251$ and $\Omega = 2.1$.)

computed by Eq. (2.61) for specified parameters in Chapter 2. The hollow circular symbols are relative to the initial and final points for grazing. The location of grazing points are marked by solid circular symbols and labeled by "Grazing".

The parameters $a = 20$, $c = 100$, $E = 1$, $d = 0.5$ and $x_i = 1$ are used for numerical simulations. Consider a motion with a single grazing first for $\Omega = 2.10$ and the initial condition $(\Omega t_i, y_i) \approx (6.1117, 6.5251)$ at $x_i = 1$. The Poincaré mapping sections through the switching planes are plotted in Fig. 8.10 for a strange attractor of chaotic motion relative to mapping structures ($P_{643(12)1}$, P_{6431} and $P_{6(45)431}$). The initial, grazing switching manifold of P_2 has three intersected points with the strange attractor in Ξ_1, as labeled by the hollow circular symbols. Based on the three intersection points, the corresponding grazing points are $(\Omega t_i, y_i) \approx (2.056, 0.0), (2.087, 0.0)$ and $(2.094, 0.0)$, as labeled by the solid circular symbols and "Grazing". The new invariant set of the strange attractor has two branches in both Ξ_1 and Ξ_2 compared to the strange attractor relative to P_{6431}. Hence, two mappings relative to $P_2 : \Xi_2 \to \Xi_1$ and one mapping pertaining to $P_1 : \Xi_1 \to \Xi_2$ exist. Namely, this chaotic motion includes the mapping structure $P_{643(12)1}$ and P_{6431}. However, in the switching section Ξ_3, the initial, grazing switching manifold of the mapping P_5 similar to mapping P_2 has three intersected points with the strange attractor. The grazing locations are near $(\Omega t_i, y_i) \approx (5.200, 0.0), (5.229, 0.0)$ and $(5.235, 0.0)$ accordingly. The initial grazing manifold of mapping P_1 is almost tangential to the strange attractor at point $(\Omega t_i, y_i) \approx (5.24, 2.93)$ in Ξ_1, so, some points in the strange attractor in both Ξ_1 and Ξ_2 are very close to the grazing point $(\Omega t_i, y_i) \approx (0.0193, 0.0)$, as shown in Fig. 8.10. Similarly, the initial grazing manifold of mapping P_3 almost tangential to the strange attractor is observed.

To observe the variation of the fragmentized attractor relative to mapping P_{6431}, the fragmentized strange attractor of chaotic motion for $\Omega = 1.98$ is presented in Fig. 8.11. The shape of the strange attractor is different from the one in Fig. 8.10. It is observed the grazing locations are near $(\Omega t_i, y_i) \approx (2.018, 0.0), (1.991, 0.0), (1.956, 0.0)$ and $(1.929, 0.0)$ on the switching sets Ξ_{12}. The grazing locations are in vicinity of $(\Omega t_i, y_i) \approx (5.159, 0.0), (5.133, 0.0), (5.098, 0.0)$ and $(5.071, 0.0)$ on the switching sets Ξ_{34}. The two fragmentized attractors are totally different. With decreasing excitation frequency, such fragmentized strange attractors of chaotic motions will disappear. Further, the symmetric and asymmetric periodic motion relative to mapping $P_{6(45)43(12)1}$ will appear. Once the grazing of the asymmetric periodic motion occurs, the fragmentized strange attractor will exist. Consider an excitation frequency $\Omega = 1.4$ with the initial condition $(\Omega t_i, y_i) \approx (0.1062, 2.4511)$ at $x_i = E$ for illustration. The Poincaré mapping sections of the strange attractor of chaotic motion are shown in Fig. 8.12. There are three branches of the strange attractor. The initial grazing manifolds of the mappings P_2 in Ξ_2 and P_5 in Ξ_4 have one intersected point with one of three branches of the strange attractor, and the corresponding grazing points are $(\Omega t_i, y_i) \approx (1.876, 0.0)$ and $(5.0375, 0.0)$ at Ξ_{12} and Ξ_{34}, respectively. Because of grazing, one of three branches of the strange attractor is generated by such a grazing. Therefore, the strange attractor of the chaotic

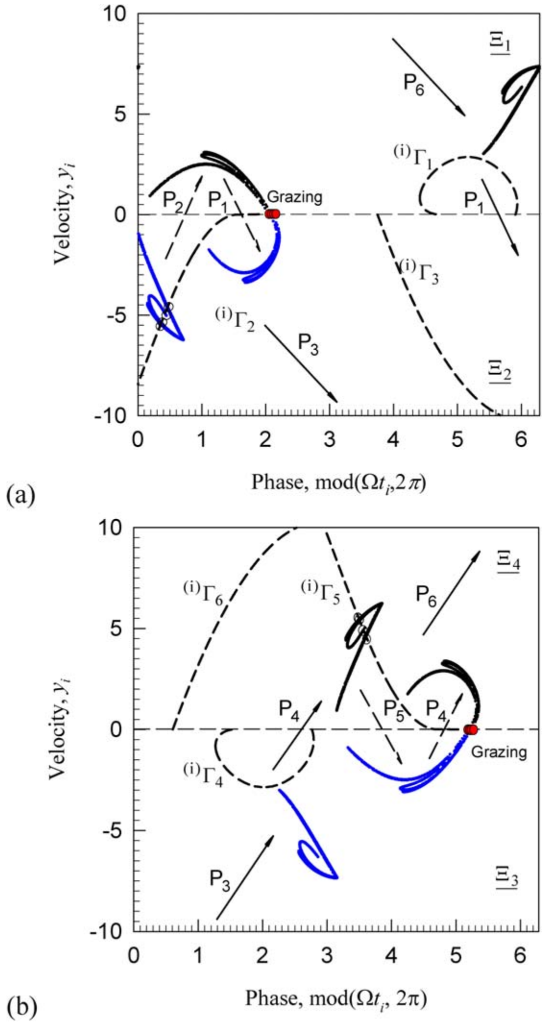


Figure 8.11. Fragmented strange attractor for chaotic motion associated with mappings $P_{6(45)431}$, P_{6431} and $P_{643(12)1}$: (a) subsets of switching plane Ξ_{12} (Ξ_1 and Ξ_2), and (b) subsets of switching plane Ξ_{34} (Ξ_3 and Ξ_4). ($a = 20$, $c = 100$, $E = 1$, $d = 0.5$, $\Omega t_i \approx 5.7257$, $x_i = 1$, $y_i \approx 6.2363$ and $\Omega = 1.89$.)

motion includes mapping structures $P_{6(45)43(12)1}$, $P_{6(45)^2 43(12)1}$ and $P_{6(45)43(12)^2 1}$. Similarly, the other fragmentized strange attractors of the chaotic motion in such a problem can be illustrated.

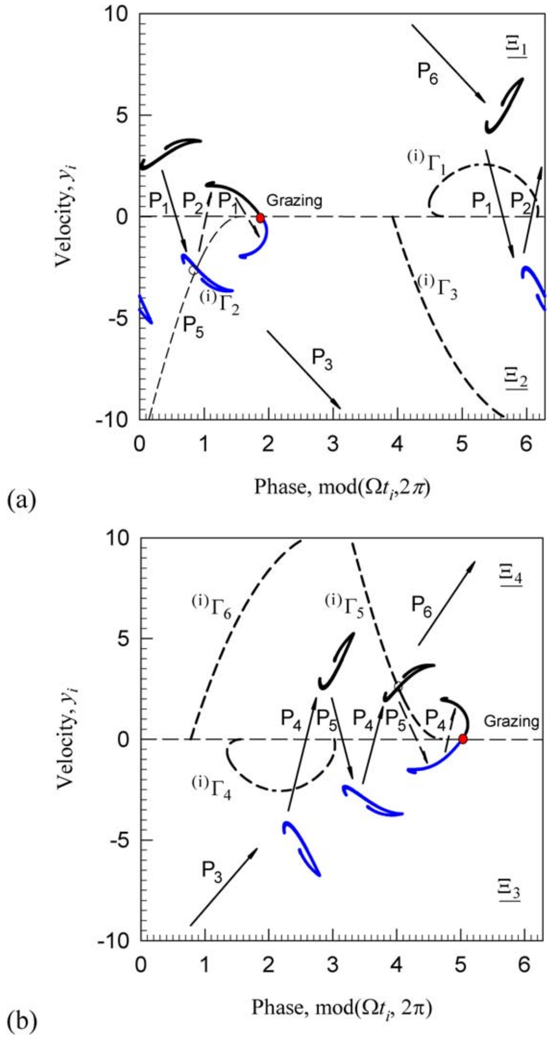


Figure 8.12. Chaotic motion relative to mappings $P_{6(45)43(12)1}$, $P_{6(45)43(12)^21}$ and $P_{6(45)^243(12)1}$: (a) subsets of switching plane Ξ_{12} (Ξ_1 and Ξ_2), and (b) subsets of switching plane Ξ_{34} (Ξ_3 and Ξ_4). ($a = 20$, $c = 100$, $E = 1$, $d = 0.5$, $\Omega t_i \approx 0.1062$, $x_i = 1$, $y_i \approx 2.4511$ and $\Omega = 1.4$.)

From the foregoing numerical simulations, it is observed that the initial and final grazing, switching manifolds are invariant for given system parameters. It is very important to determine the mechanism of the strange attractor fragmentation. The criteria and topological structure for the fragmentation of the strange attractor need to be further developed as in hyperbolic strange attractors. The fragmentation of the strange attractors extensively exists in discontinuous dynamical systems, which will help us better understand chaotic motions in discontinuous dynamic systems.

APPENDIX A

With an initial condition $(t, x, \dot{x}) = (t_i, x_i, y_i)$, the general solutions for linear systems are given as follows.

Case I: $d_j^2 - c_j > 0$

$$\begin{aligned} x &= \mathrm{e}^{-d_j(t-t_i)}\big(C_1^{(j)}\mathrm{e}^{\lambda_d^{(j)}(t-t_i)} + C_2^{(j)}\mathrm{e}^{-\lambda_d^{(j)}(t-t_i)}\big) \\ &\quad + D_1^{(j)}\cos\Omega t + D_2^{(j)}\sin\Omega t + D_0^{(j)}, \\ \dot{x} &= \mathrm{e}^{-d_j(t-t_i)}\big[\big(\lambda_d^{(j)} - d_j\big)C_1^{(j)}\mathrm{e}^{\lambda_d(t-t_i)} - \big(\lambda_d^{(j)} + d_j\big)C_2^{(j)}\mathrm{e}^{-\lambda_d^{(j)}(t-t_i)}\big] \\ &\quad - D_1^{(j)}\Omega\sin\Omega t + D_2^{(j)}\Omega\cos\Omega t \end{aligned} \tag{A.1}$$

where

$$\begin{aligned} C_1^{(j)} &= \frac{1}{2\omega_d^{(j)}}\Big\{\dot{x}_i + \big(x_i - D_0^{(j)}\big)\big(d_j + \omega_d^{(j)}\big) \\ &\quad - \big[D_1^{(j)}\big(d_j + \omega_d^{(j)}\big) + D_2^{(j)}\Omega\big]\cos\Omega t_i \\ &\quad + \big[D_1^{(j)}\Omega - D_2^{(j)}\big(d_j + \omega_d\big)\big]\sin\Omega t_i\Big\}, \\ C_2^{(j)} &= \frac{1}{2\omega_d^{(j)}}\Big\{-\dot{x}_i + \big(x_i - D_0^{(j)}\big)\big(-d_j + \omega_d^{(j)}\big) \\ &\quad - \big[D_1^{(j)}\Omega + D_2^{(j)}\big(-d_j + \omega_d^{(j)}\big)\big]\sin\Omega t_i \\ &\quad - \big[D_1^{(j)}\big(-d_j + \omega_d^{(j)}\big) - D_2^{(j)}\Omega\big]\cos\Omega t_i\Big\}; \end{aligned} \tag{A.2}$$

$$\begin{aligned} D_0^{(j)} &= -\frac{b_j}{c_j}, \qquad D_1^{(j)} = \frac{a(c_j - \Omega^2)}{(c_j - \Omega^2)^2 + (2d_j\Omega)^2}, \\ D_2^{(j)} &= \frac{a(2d_j\Omega)}{(c_j - \Omega^2)^2 + (2d_j\Omega)^2}, \qquad \lambda_d^{(j)} = \sqrt{d_j^2 - c_j}. \end{aligned} \tag{A.3}$$

Case II: $d_j^2 - c_j < 0$

$$\begin{aligned} x &= \mathrm{e}^{-d_j(t-t_i)}\big[C_1^{(j)}\cos\omega_d^{(j)}(t-t_i) + C_2^{(j)}\sin\omega_d^{(j)}(t-t_i)\big] \\ &\quad + D_1^{(j)}\cos\Omega t + D_2^{(j)}\sin\Omega t + D_0^{(j)}, \\ \dot{x} &= \mathrm{e}^{-d_j(t-t_i)}\big[\big(C_2^{(j)}\omega_d^{(j)} - d_jC_1^{(j)}\big)\cos\omega_d^{(j)}(t-t_i) \\ &\quad - \big(C_1^{(j)}\omega_d^{(j)} + d_jC_2^{(j)}\big)\sin\omega_d^{(j)}(t-t_i)\big] \\ &\quad - D_1^{(j)}\Omega\sin\Omega t + D_2^{(j)}\Omega\cos\Omega t \end{aligned} \tag{A.4}$$

where

$$\begin{aligned}
C_1^{(j)} &= x_i - D_1^{(j)} \cos \Omega t_i - D_2^{(j)} \sin \Omega t_i - D_0^{(j)}, \\
C_2^{(j)} &= \frac{1}{\omega_d^{(j)}} \big[d_j \big(x_i - D_1^{(j)} \cos \Omega t_i - D_2^{(j)} \sin \Omega t_i - D_0^{(j)} \big) \\
&\quad + \dot{x}_i + D_1^{(j)} \Omega \sin \Omega t_i - D_2^{(j)} \Omega \cos \Omega t_i \big], \\
\omega_d^{(j)} &= \sqrt{c_j - d_j^2}.
\end{aligned} \tag{A.5}$$

Case III: $d_j^2 - c_j = 0$

$$\begin{aligned}
x &= \mathrm{e}^{-d_j(t-t_i)} \big[C_1^{(j)}(t - t_i) + C_2^{(j)} \big] + D_1^{(j)} \cos \Omega t + D_2^{(j)} \sin \Omega t + D_1^{(j)}, \\
\dot{x} &= \mathrm{e}^{-d_j(t-t_i)} \big[C_1^{(j)} - C_1^{(j)} d_j (t - t_i) - d_j C_2^{(j)} \big] - D_1^{(j)} \Omega \sin \Omega t \\
&\quad + D_2^{(j)} \Omega \cos \Omega t
\end{aligned} \tag{A.6}$$

where

$$\begin{aligned}
C_2^{(j)} &= x_i - D_1^{(j)} \cos \Omega t_i - D_2^{(j)} \sin \Omega t_i - D_0^{(j)}, \\
C_1^{(j)} &= x_i + \cos \Omega t_i \big(D_2^{(j)} \Omega - d_j D_1^{(j)} \big) - \sin \Omega t_i \big(D_1^{(j)} \Omega + d_j D_2^{(j)} \big) \\
&\quad - d_j D_1^{(j)}.
\end{aligned} \tag{A.7}$$

Case IV: $d_j \neq 0,\ c_j = 0$

$$\begin{aligned}
x &= C_1^{(j)} \mathrm{e}^{-2d_j(t-t_i)} + D_1^{(j)} \cos \Omega t + D_2^{(j)} \sin \Omega t + D_0^{(j)} t + C_2^{(j)}, \\
\dot{x} &= -2 d_j C_1^{(j)} \mathrm{e}^{-2d_j(t-t_i)} - D_1^{(j)} \Omega \sin \Omega t + D_2^{(j)} \Omega \cos \Omega t + D_0^{(j)}
\end{aligned} \tag{A.8}$$

where

$$\begin{aligned}
C_1^{(j)} &= -\frac{1}{2d_j} \big(\dot{x}_i + D_1^{(j)} \Omega \sin \Omega t_i - D_2^{(j)} \Omega \cos \Omega t_i - D_0^{(j)} \big), \\
C_2^{(j)} &= \frac{1}{2d_j} \big[2 d_j x_i + \dot{x}_i + \big(D_1^{(j)} \Omega - 2 d_j D_2^{(j)} \big) \sin \Omega t_i \\
&\quad - \big(2 d_j D_1^{(j)} + D_2^{(j)} \Omega \big) \cos \Omega t_i - 2 d_j D_0^{(j)} t_i - D_0^{(j)} \big].
\end{aligned} \tag{A.9}$$

Case V: $d_j = 0,\ c_j = 0$

$$\begin{aligned}
x &= -\frac{a}{\Omega^2} \cos \Omega t - \frac{1}{2} b_j t^2 + C_1^{(j)} t + C_2^{(j)}, \\
\dot{x} &= \frac{a}{\Omega} \sin \Omega t - b_j t + C_1^{(j)}
\end{aligned} \tag{A.10}$$

where

$$C_1^{(j)} = \dot{x}_i - \frac{a}{\Omega}\sin\Omega t_i + b_j t_i,$$
$$C_2^{(j)} = x_i - \dot{x}t_i + \frac{a}{\Omega^2}\cos\Omega t_i + \frac{a}{\Omega}t_i\sin\Omega t_i - \frac{1}{2}b_j t_i^2. \tag{A.11}$$

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